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Fredholm theory and localized SVEP

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Abstract.
The purpose of this work is to provide a streamlined approach to some classical results of Fredholm theory, together an extension of these, by using some tools of local spectral theory, in particular through a localized version of the single-valued extension property.

Introduction

The contents may be thought as a first orientation and the material is expository in the style, most of the results are given without proof, but are appropriately referred by giving suitable references. The material has been arranged in six sections:

- Section 1: The single-valued extension property.
- Section 2: Classes of operators in Fredholm theory.
- Section 3: The localized SVEP and Fredholm theory.
- Section 4: The localized SVEP under commuting perturbations.
- Section 5: Weyl and Browder spectra under perturbations.
- Section 6: Polaroid type operators.

It is clear from the organization of this work, that our main purpose is of showing the interaction between the Fredholm theory and the single-valued extension property. Some of the results established are very new.

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1. Single-valued extension property

The single-valued extension property, and more generally, the local spectral theory, arises from the substantial attempts to transfer some of the important properties of the spectral theory of normal operators on Hilbert spaces to the more general setting of Banach spaces. The early studies of local spectral theory were initiated by Dunford, and treated in a more systematic way in the monographs by Dunford and Schwartz [46], and I. Colojoară, C. Foiaş [38]. In the last years an increasing role in local spectral theory, especially in connection with Fredholm theory, has been assumed by a localized version of the single-valued extension property. To introduce this property we first introduce some typical tools of local spectral theory, and in order to give a first motivation, we begin with some considerations on spectral theory. The *spectrum* of $T \in L(X)$, where $L(X)$ denotes the Banach algebra of all bounded linear operators on a complex infinite dimensional Banach space, is defined as

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bijective}\}.$$

It is well-known that the spectrum is a compact subset of $\mathbb{C}$ and $\sigma(T) = \sigma(T^*)$ for all $T \in L(X)$. Since X is a complex Banach space then every $T \in L(X)$ has non-empty spectrum. The complement $\rho(T) := \mathbb{C} \setminus \sigma(T)$ is called the *resolvent* of T. It is well known that the resolvent function $R(\lambda, T) := (\lambda I - T)^{-1}$ of $T \in L(X)$ is an analytic operator-valued function defined on the resolvent set $\rho(T)$. Evidently, the vector-valued analytic function $f_x : \rho(T) \to X$ defined as

$$f_x(\lambda) := R(\lambda, T)x \quad \text{for any } x \in X,$$

satisfies the equation

$$(\lambda I - T)f_x(\lambda) = x \ \text{ for all } \lambda \in \rho(T). \tag{1}$$

Suppose that $T \in L(X)$ has an isolated point λ_0. Let P_0 denote the spectral projection of T associated with λ_0 defined by the classical functional calculus. Then the spectrum of the restriction $T_0 := T|P_0(X)$ is $\{\lambda_0\}$, thus $\lambda I - T_0$ is invertible for all $\lambda \neq \lambda_0$. Let $x \in P_0(X)$. Obviously, the equation (1) has the analytic solution

$$g_x(\lambda) := (\lambda I - T_0)^{-1}x \ \text{ for all } \lambda \in \mathbb{C} \setminus \{\lambda_0\}.$$

This shows that it is possible to find analytic solutions of the equation $(\lambda I - T)f_x(\lambda) = x$ for some, and sometimes even for all, values of λ that are in the spectrum of T.

These considerations property lead the following concept:

Definition 1.1. *Given an arbitrary operator $T \in L(X)$, X a Banach space, let $\rho_T(x)$ denote the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood $\mathcal{U}_\lambda$ of λ in $\mathbb{C}$ and an analytic function $f : \mathcal{U}_\lambda \to X$ such that the equation*

$$(\mu I - T)f(\mu) = x \quad \textit{holds for all } \mu \in \mathcal{U}_\lambda. \tag{2}$$

If the function f is defined on the set $\rho_T(x)$ then it is called a local resolvent function *of T at x. The set $\rho_T(x)$ is called the* local resolvent *of T at x. The* local spectrum *$\sigma_T(x)$ of T at the point $x \in X$ is defined to be the set $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$.*

Evidently $\rho_T(x)$ is the open subset of $\mathbb{C}$ given by the union of the domains of all the local resolvent functions. Moreover,

$$\rho(T) \subseteq \rho_T(x) \quad \text{ and } \quad \sigma_T(x) \subseteq \sigma(T).$$

It is immediate to check the following elementary properties of $\sigma_T(x)$:

(a) $\sigma_T(0) = \emptyset$;

(b) $\sigma_T(\alpha x + \beta y) \subseteq \sigma_T(x) \cup \sigma_T(y)$ for all $x, y \in X$;

(c) For every $F \subseteq \mathbb{C}$, $\sigma_{\lambda I+T}(x) \subseteq F$ if and only if $\sigma_T(x) \subseteq F - \lambda$, In particular, $\sigma_{(\lambda I-T)}(x) \subseteq \{0\}$ if and only if $\sigma_T(x) \subseteq \{\lambda\}$.

(d) If $T, S \in L(X)$ commutes then $\sigma_T(Sx) \subseteq \sigma_T(x)$.

It is well known that given two operators $R, S \in L(X)$, the spectra $\sigma(RS)$ and $\sigma(SR)$ may be different only at 0. The following theorem, due to Benhida and Zerouali [23] shows a local spectral version of this result.

Theorem 1.2. *Let X and Y be Banach spaces and consider two operators $S \in L(X, Y)$ and $R \in L(Y, X)$. Then we have:*

(i) *For every $x \in X$ the following inclusions hold:*

$$\sigma_{SR}(Sx) \subseteq \sigma_{RS}(x) \subseteq \sigma_{SR}(Sx) \cup \{0\}.$$

If S is injective then $\sigma_{RS}(x) = \sigma_{SR}(Sx)$ for all $x \in X$.

(ii) *For every $y \in Y$ the following inclusions hold:*

$$\sigma_{RS}(Ry) \subseteq \sigma_{SR}(y) \subseteq \sigma_{RS}(Ry) \cup \{0\}.$$

If R is injective then $\sigma_{RS}(Ry) = \sigma_{SR}(y)$ for all $y \in Y$.

Proof. (i) Let $\lambda \notin \sigma_{RS}(x)$ and $f : \mathcal{U} \to X$ an analytic function defined in a neighborhood $\mathcal{U}$ of λ such that $(\mu I - RS)f(\mu) = x$ for all $\mu \in \mathcal{U}$. Then $Sx = (\mu I - SR)Sf(\mu) = Sx$, hence $\lambda \notin \sigma_{SR}(Sx)$, so the first inclusion in (i) is proved. To show the second inclusion, let $\lambda \notin \sigma_{SR}(Sx) \cup \{0\}$ and denote by $g(\mu)$ an Y-valued analytic function defined on a neighborhood $\mathcal{U}$ of λ such that $(\mu I - SR)g(\mu) = Sx$ for all $\mu \in \mathcal{U}$. If we set $h(\mu) := \frac{1}{\mu}(x - Rg(\mu))$ it is easy to check that $(\mu I - RS)h(\mu) = x$, so $\lambda \notin \sigma_{RS}(x)$.

To show the second statement, assume that $\lambda \notin \sigma_{SR}(Sx)$. There is no harm if we assume $\lambda = 0$. Thus, assume $0 \notin \sigma_{SR}(Sx)$. and let $g(\mu)$ an Y-valued analytic function defined in a neighborhood $\mathcal{U}$ of 0 such that $(\mu I - SR)g(\mu) = Sx$. For $\mu = 0$ we have $-SRg(0) = Sx$ and from the injectivity of S it follows that $x = Rg(0)$. Moreover,

$$\mu g(\mu) = Sx + SRg(\mu) = S(x + Rg(\mu))$$

so $g(\mu) = S[\frac{1}{\mu}(x + Rg(\mu)]$. Note that

$$SRg'(0) = \lim_{\mu \to 0} \frac{SRg(\mu) - SRg(0)}{\mu} = \lim_{\mu \to 0} \frac{SRg(\mu) + Sx}{\mu} = \lim_{\mu \to 0} g(\mu) = g(0).$$

Set

$$h(\mu) := \begin{cases} \dfrac{1}{\mu}(x - Rg(\mu)) & \text{if } \mu \neq 0, \\ Rg'(0) & \text{if } \mu = 0, \end{cases}$$

We have $S[(\mu I - RS)h(\mu) - x] = 0$ for all U. Indeed, we have seen in the first part of the proof that for $\mu \neq 0$ we have $(\mu I - RS)h(\mu) - x = 0$, while for $\mu = 0$ we have

$$S[-RSRg'(0) - x] = -SR(SRg'(0) - Sx = -SRg(0) - Sx = Sx - Sx = 0.$$

Since S is injective then we have $(\mu I - RS)h(\mu) = x$ for all $\mu \in U$, hence $0 \notin \sigma_{RS}(x)$.

(ii) The proof is analogous. ∎

Takeing $S = T$ and $R = I$ in Theorem 1.2, we obtain

Corollary 1.3. *Let $T \in L(X)$ and $x \in X$. Then we have*

(i) $\sigma_T(Tx) \subseteq \sigma_T(x) \subseteq \sigma_T(Tx) \cup \{0\}$.

(ii) *If T is injective then $\sigma_T(Tx) = \sigma_T(x)$.*

The following example shows that if S in not injective we may have $\sigma_{RS}(x) \neq \sigma_{SR}(Sx)$.

Example 1.4. Let S denote the shift operator defined in the usual Hardy space H and $R := S^*$ the adjoint of S. Then, RS is the identity operator, while SR is the projection of X onto the range $S(H)$. In particular, $\sigma_{RS}(x) = \{1\}$ for all $0 \neq x \in H$, $\sigma_{SR}(x) = \{1\}$ if $x \in S(H)$, $\sigma_{SR}(x) = \{0\}$ if $x \in \ker R$ and $\sigma_{SR}(x) = \{0, 1\}$ otherwise. In this case, $\sigma_{RS}(Sx)$ is strictly contained in $\sigma_{SR}(x)$.

Definition 1.5. *Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have* the single-valued extension property *at $\lambda_0 \in \mathbb{C}$, abbreviated T has the SVEP at λ_0, if for every neighborhood $\mathcal{U}$ of λ_0 the only analytic function $f : \mathcal{U} \to X$ which satisfies the equation*

$$(\lambda I - T)f(\lambda) = 0$$

is the constant function $f \equiv 0$.

The operator T is said to have the SVEP if T has the SVEP at every $\lambda \in \mathbb{C}$.

Remark 1.6. In the sequel we collect some basic properties of the SVEP.

(a) The SVEP ensures the consistency of the local solutions of the equation (2), in the sense that if $x \in X$ and T has the SVEP at $\lambda_0 \in \rho_T(x)$ then there exists a neighborhood $\mathcal{U}$ of λ_0 and an *unique* analytic function $f : \mathcal{U} \to X$ satisfying the equation $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathcal{U}$.

The SVEP also ensures the existence of a maximal analytic extension $\tilde{f}$ of $R(\lambda, T)x := (\lambda I - T)^{-1}x$ to the set $\rho_T(x)$ for every $x \in X$. This function identically verifies the equation

$$(\mu I - T)\tilde{f}(\mu) = x \quad \text{for every } \mu \in \rho_T(x)$$

and, obviously,

$$\tilde{f}(\mu) = (\mu I - T)^{-1}x \quad \text{for every } \mu \in \rho(T).$$

(b) It is immediate to verify that the SVEP is inherited by the restrictions on invariant closed subspaces. Moreover,

$$\sigma_T(x) \subseteq \sigma_{T|M}(x) \quad \text{for every } x \in M.$$

(c) Obviously, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$. From the identity theorem for analytic function it easily follows that an operator always has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, T has SVEP at every isolated point of the spectrum $\sigma(T)$. Since $\sigma(T^*) = \sigma(T)$, where T^* denotes the dual of T, it then follows that also T^* has SVEP at every isolated point of the spectrum $\sigma(T)$.

(d) Let $\sigma_{\mathrm{p}}(T)$ denote the *point spectrum* of $T \in L(X)$, i.e.,

$$\sigma_{\mathrm{p}}(T) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\}.$$

It is easy to see that if $\sigma_{\mathrm{p}}(T)$ has empty interior then T has SVEP, in particular every operator with real spectrum has SVEP. A rather immediate argument shows the following implication:

$$\sigma_{\mathrm{p}}(T) \text{ does not cluster at } \lambda_0 \Rightarrow T \text{ has the SVEP } \lambda_0.$$

Indeed, suppose that $\sigma_p(T)$ does not cluster at λ_0. Then there is an open neighborhood $\mathcal{U}$ $\mathcal{U}$ of λ_0 such that $\lambda I - T$ is injective for every $\lambda \in \mathcal{U}$, $\lambda \neq \lambda_0$. Let $f : \mathcal{V} \to X$ be an analytic function defined on another neighborhood $\mathcal{V}$ of λ_0 such that the equation $(\lambda I - T)f(\lambda) = 0$ holds for every $\lambda \in \mathcal{V}$. We may assume that $\mathcal{V} \subseteq \mathcal{U}$. Then $f(\lambda) \in \ker(\lambda I - T) = \{0\}$ for every $\lambda \in \mathcal{V}$, $\lambda \neq \lambda_0$, so $f(\lambda) = 0$ for every $\lambda \in \mathcal{V}$, $\lambda \neq \lambda_0$. Since f is continuous at λ_0 we then conclude that $f(\lambda_0) = 0$. Hence $f \equiv 0$ in $\mathcal{V}$ and therefore T has the SVEP at λ_0.

Observe that T may have SVEP, although $\sigma_{\mathrm{p}}(T) \neq \emptyset$. For instance, if $X := B(\Omega)$ the Banach algebra of all bounded complex-valued functions on a compact Hausdorff space Ω, endowed with pointwise operations and supremum norm, the operator $T \in L(X)$, defined by the assignment

$$(Tf)(\lambda) := \lambda f(\lambda) \quad \text{for all } \lambda \in \Omega,$$

has $\sigma_p(T) \neq \emptyset$, while T has SVEP since the ascent $p(\mu I - T) \leq 1$ for all $\mu \in \mathbb{C}$, and this, as we shall see later, entails SVEP.

(e) The SVEP is transmitted under translations, i.e. $T \in L(X)$ has SVEP if and only if $\lambda I - T$ has SVEP.

(f) Let $x \in X$ and $\mathcal{U}$ an open subset of $\mathbb{C}$. Suppose that $f : \mathcal{U} \to X$ is an analytic function for which $(\mu I - T)f(\mu) = x$ for all $\mu \in \mathcal{U}$. Then $\mathcal{U} \subseteq \rho_T(f(\lambda))$ for all $\lambda \in \mathcal{U}$. Moreover,

$$\sigma_T(x) = \sigma_T(f(\lambda)) \quad \text{for all } \lambda \in \mathcal{U}. \tag{3}$$

Now we introduce an important class of subspaces which play an important role in local theory.

Definition 1.7. *For every subset F of $\mathbb{C}$ the* local spectral subspace *of an operator $T \in L(X)$ associated with F is the set*

$$X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}.$$

Obviously, if $F_1 \subseteq F_2 \subseteq \mathbb{C}$ then $X_T(F_1) \subseteq X_T(F_2)$ and obviously $X_T(F) = X_T(F \cap \sigma(T))$. Indeed $X_T(F) \cap \sigma(T)) \subseteq X_T(F)$. Conversely, if $x \in X_T(F)$ then $\sigma_T(x) \subseteq F \cap \sigma(T)$, and hence $x \in X_T(F \cap \sigma(T))$. Moreover, it is easily seen from the basic properties of the local spectrum that $X_{\lambda I+T}(F) = X_T(F - \lambda)$. Further basic properties of local spectral subspaces are collected in the sequel (see [68, Chap.1] for a proof) .

Theorem 1.8. *Let $T \in L(X)$ and F every subset of $\mathbb{C}$. Then the following properties hold:*

(i) *$X_T(F)$ is a linear hyper-invariant subspace for T, i.e., for every bounded operator S such that $TS = ST$ we have $S(X_T(F)) \subseteq X_T(F)$;*

(ii) *If $\lambda \notin F$, then $(\lambda I - T)(X_T(F)) = X_T(F)$;*

(iii) *Suppose that $\lambda \in F$ and $(\lambda I - T)x \in X_T(F)$ for some $x \in X$. Then $x \in X_T(F)$;*

(iv) *For every family $(F_j)_{j\in J}$ of subsets of $\mathbb{C}$ we have*

$$X_T(\bigcap_{j\in J} F_j) = \bigcap_{j\in J} X_T(F_j);$$

(v) *If Y is an invariant closed subspace for T such that $\sigma(T|Y) \subseteq F$, then $Y \subseteq X_T(F)$. In particular, $Y \subseteq X_T(\sigma(T|Y))$ holds for every closed T-invariant closed subspace of X.*

(vi) *ker $(\lambda I - T)^n \subseteq X_T(\{\lambda\}$ for all $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$;*

(vii) *If $T \in L(X)$ has the SVEP and F_1 and F_2 are two closed and disjoint subsets of $\mathbb{C}$ then*

$$X_T(F_1 \cup F_2) = X_T(F_1) \oplus X_T(F_2),$$

where the direct sum is in the algebraic sense.

We have already observed that 0 has an empty local spectrum shows that if T has the SVEP then 0 is the *unique* element of X having empty local spectrum.

Theorem 1.9. *[68, Prop.1.2.16]If $T \in L(X)$ the following statements are equivalent :*

(i) *T has the SVEP;*

(ii) *$X_T(\emptyset) = \{0\}$;*

(iii) *$X_T(\emptyset)$ is closed.*

We now introduce a somewhat variant of the concept of analytic subspace $X_T(\Omega)$. These subspaces are more appropriate for certain general questions of local spectral theory than the analytic subspace $X_T(F)$, and in particular these subspaces are more useful if T does not have SVEP.

Definition 1.10. *Let $F \subseteq \mathbb{C}$ be a closed subset. If $T \in L(X)$ the* glocal local subspace *$\mathcal{X}_T(F)$ is defined as the set of all $x \in X$ such that there is an analytic function $f : \mathbb{C} \setminus F \to X$ such that*

$$(\lambda I - T)f(\lambda) = x \quad \textit{for all } \lambda \in \mathbb{C} \setminus F.$$

It is easy to verify that $\mathcal{X}_T(F)$ is a linear subspace of X. Clearly

$$\mathcal{X}_T(F) \subseteq X_T(F) \quad \text{for every closed subset } F \subseteq \mathbb{C}\,. \tag{4}$$

In the following theorem we show few basic properties of the glocal subspaces. Some of these properties are rather similar to those of local spectral subspaces. The interested reader may be found further results on glocal spectral subspaces in Laursen and Neumann [68].

Theorem 1.11. *For an operator $T \in L(X)$, the following statements hold:*

(i) $\mathcal{X}_T(\emptyset) = \{0\}$ *and* $\mathcal{X}_T(\sigma(T)) = X$*;*

(ii) $\mathcal{X}_T(F) = \mathcal{X}_T(F \cap \sigma(T))$ *and* $(\lambda I - T)\mathcal{X}_T(F) = \mathcal{X}_T(F)$ *for every closed set* $F \subseteq \mathbb{C}$ *and all* $\lambda \in \mathbb{C} \setminus F$*;*

(iii) *If* $(\lambda I - T)x \in \mathcal{X}_T(F)$ *for some* $\lambda \in F$*, then* $x \in \mathcal{X}_T(F)$*.*

(iv) $\mathcal{X}_T(F_1 \cup F_2) = \mathcal{X}_T(F_1) + \mathcal{X}_T(F_2)$ *for all disjoint closed subsets* F_1 *and* F_2 *of* $\mathbb{C}$*;*

(v) *T has the SVEP if and only if* $\mathcal{X}_T(F) = X_T(F)$*, for every closed subset* $F \subseteq \mathbb{C}$*, and this happens if and only if* $\mathcal{X}_T(F) \cap \mathcal{X}_T(G)$ *for all disjoint closed subsets F and G of* $\mathbb{C}$*.*

(vi) *Let* $\sigma(T) = F_1 \oplus F_2$*, where* F_1 *and* F_2 *are disjoint closed subsets of* $\mathbb{C}$*. Then the subspaces* $\mathcal{X}_T(F_i)$*,* $i = 1, 2$*, are closed and* $X = \mathcal{X}_T(F_1) \oplus \mathcal{X}_T(F_2)$*.*

A very important class of operators is given by the class of all *bounded below* operators, i.e. the injective operators having closed range. A classical result shows that the properties to be bounded below or to be surjective are dual each other. Two important parts of the spectrum $\sigma(T)$ are defined as follows: The *approximate point spectrum* of $T \in L(X)$ defined as

$$\sigma_{\mathrm{ap}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}$$

and the *the surjectivity spectrum* of T defined as

$$\sigma_{\mathrm{s}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is onto.}\}$$

These two spectra are non-empty closed subset of $\sigma(T)$ and

$$\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{s}}(T^*) \quad \text{and} \quad \sigma_{\mathrm{ap}}(T^*) = \sigma_{\mathrm{s}}(T).$$

Evidently, $\mathcal{X}_T(F) = X$ for a closed subset F of $\mathbb{C}$ implies that $\sigma_{\mathrm{s}}(T) \subseteq F$. The next result shows that reverse implication holds. This result is based on a deep result of Leiterer [70] and we refer to [68, Theorem 3.3.12] for a proof.

Theorem 1.12. *If $T \in L(X)$ and $F \subseteq \mathbb{C}$ is closed then the following assertions hold:*

(i) $\mathcal{X}_T(F) = X$ *if and only if* $\sigma_{\mathrm{s}}(T) \subseteq F$*.*

(ii) $\mathcal{X}_T(F) = \{0\}$ *if and only if* $\sigma_{\mathrm{ap}}(T) \cap F = \emptyset$*.*

The next theorem shows that the glocal spectral subspaces behave canonically under the functional calculus.

Theorem 1.13. *[68, Theorem 3.3.6]) If $T \in L(X)$, and $f : \mathcal{U} \to X$ is analytic on an open neighborhood of the spectrum $\sigma(T)$ then*

$$\mathcal{X}_{f(T)}(F) = \mathcal{X}_T(f^{-1}(F)) \quad \textit{for all closed subsets } F \subseteq \mathbb{C}.$$

The glocal spectral subspace $\mathcal{X}_T(\mathbf{D}(0, \varepsilon))$ associated with the closed disc $\mathbf{D}(0, \varepsilon)$ may be characterized as follows, see [1, Theorem 2.20] :

Theorem 1.14. *If $T \in L(X)$ then*

$$\mathcal{X}_T(\mathbf{D}(0, \varepsilon)) = \left\{ x \in X : \limsup_{n \to \infty} \|T^n x\|^{1/n} \leq \varepsilon \right\}. \tag{5}$$

We now introduce an important subspace in Fredholm theory.

Definition 1.15. *The* quasi-nilpotent part $H_0(T)$ *of an operator* $T \in L(X)$ *is defined as* $H_0(T) = \mathcal{X}_T(\{0\})$.

The quasi-nilpotent part of an operator may be characterized as follows:

Corollary 1.16. *For every* $T \in L(X)$ *we have*

$$H_0(T) = \{x \in X : \limsup_{n\to\infty} \|T^n x\|^{1/n} = 0\}. \tag{6}$$

Moreover, if T has SVEP then $H_0(T) = X_T(\{0\})$.

Proof. Clearly, the equality (6) is obtained by taking $\varepsilon = 0$ in Theorem 1.14. If T has SVEP then $\mathcal{X}_T(\{0\}) = X_T(\{0\})$, by part (iv) of Theorem 1.11. ∎

The following example shows that, the quasi-nilpotent part $H_0(T)$ need not be closed, also if T has SVEP.

Example 1.17. Let $X := \ell_2 \oplus \ell_2 \cdots$ be provided with the norm

$$\|x\| := \left(\sum_{n=1}^{\infty} \|x_n\|^2\right)^{1/2} \quad \text{for all } x := (x_n) \in X,$$

and define

$$T_n e_i := \begin{cases} e_{i+1} & \text{if } i = 1, \cdots, n\,, \\ \dfrac{e_{i+1}}{i-n} & \text{if } i > n. \end{cases}$$

It is easily seen that

$$\|T_n^{n+k}\| = 1/k! \quad \text{and} \quad (1/k!)^{1/n+k} \text{ as } k \to \infty.,$$

from which we obtain that $\sigma(T_n) = \{0\}$. Moreover, every T_n is injective and the point spectrum $\sigma_{\mathrm{p}}(T_n) = \emptyset$, thus T_n has the SVEP.

Let us define $T := T_1 \oplus \cdots \oplus T_n \oplus \cdots$. From the estimate $\|T_n\| = 1$ for every $n \in \mathbb{N}$, we easily obtain $\|T\| = 1$. Moreover, since $\sigma_{\mathrm{p}}(T_n) = \emptyset$ for every $n \in \mathbb{N}$, it also follows that $\sigma_{\mathrm{p}}(T) = \emptyset$, hence T has SVEP.

Consider the sequence $x = (x_n) \subset X$ defined by $x_n := e_1/n$ for every n. We have

$$\|x\| = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} < \infty\,,$$

which implies that $x \in X$. Moreover,

$$\|T^n x\|^{1/n} \geq \|T_n^n \frac{e_1}{n}\|^{1/n} = (1/n)^{1/n}$$

and the last term does not converge to 0. From this it follows that $\sigma_T(x)$ contains properly $\{0\}$ and therefore, $x \notin X_T(\{0\}) = H_0(T)$.
Finally,

$$\ell_2 \oplus \ell_2 \cdots \oplus \ell_2 \oplus \{0\} \cdots \subset H_0(T),$$

where the non-zero terms are n. This holds for every $n \in \mathbb{N}$, so $H_0(T)$ is dense in X. Since $H_0(T) \neq X$ it then follows that $H_0(T)$ is not closed.

In the following we collect some basic properties of $H_0(T)$.

Lemma 1.18. *[1, Lemma 1.67] For every $T \in L(X)$, X a Banach space, we have:*

(i) $\ker(T^m) \subseteq \mathcal{N}^\infty(T) \subseteq H_0(T)$ *for every* $m \in \mathbb{N}$;

(ii) $x \in H_0(T) \Leftrightarrow Tx \in H_0(T)$;

(iii) $\ker(\lambda I - T) \cap H_0(T) = \{0\}$ *for every* $\lambda \neq 0$;

(iv) $T \in L(X)$ *is quasi-nilpotent if and only if* $H_0(T) = X$.

The following subspace has been introduced by Vrbová [97] and studied in several papers by Mbekhta ([73], [72], [75]. This subspace is, in a certain sense, the analytic counterpart of the *algebraic core* $C(T)$ which is defined as the greatest linear subspace F for which $T(F) = F$, see [1, Chap. 1] .

Definition 1.19. *Let X be a Banach space and $T \in L(X)$. The* analytical core *of T is the set $K(T)$ of all $x \in X$ such that there exists a sequence $(u_n) \subset X$ and a constant $\delta > 0$ such that:*

(1) $x = u_0$, *and* $Tu_{n+1} = u_n$ *for every* $n \in \mathbb{Z}_+$;

(2) $\|u_n\| \leq \delta^n \|x\|$ *for every* $n \in \mathbb{Z}_+$.

It is easily seen that $K(T)$ is a linear subspace of X and $T(K(T)) = K(T)$. Moreover, if $\lambda \neq 0$ then $\ker(\lambda I - T) \subseteq K(T)$. Observe that in general $K(T)$is not closed.

Theorem 1.20. *[1, Theorem 1.22] Suppose that $T \in L(X)$. If F is a closed subspace of X such that $T(F) = F$ then $F \subseteq K(T)$.*

The next result, owed to Vrbová [97] and Mbekhta [72], shows that the analytical core $K(T)$ is the local spectral subspace associated to the set $\mathbb{C} \setminus \{0\}$.

Theorem 1.21. *[1, Theorem 2.18] For every $T \in L(X)$ we have*

$$K(T) = X_T(\mathbb{C} \setminus \{0\}) = \{x \in X : 0 \notin \sigma_T(x)\}.$$

Definition 1.22. *A bounded operator $T \in L(X)$, X a Banach space, is said to be a* semi-regular *if T has closed range $T(X)$ and* $\ker T \subseteq T^n(X)$ *for every* $n \in \mathbf{N}$.

Clearly, bounded below as well as surjective operators are semi-regular. Note that the product of two semi-regular operators, also commuting semi-regular operators, need not be semi-regular. On the other hand, if $T, S \in L(X)$ are two commuting operators such that TS semi-regular, then both T and S are semi-regular. $T \in L(X)$ is semi-regular if and only if $T^\star \in L(X^*)$ is semi-regular. The *reduced minimal modulus* of T is defined to be

$$\gamma(T) := \inf_{x \notin \ker T} \frac{\|Tx\|}{\operatorname{dist}(x, \ker T)}.$$

Theorem 1.23. *[1, Theorem 1.31] Let $T \in L(X)$ be semi-regular. Then $\lambda I - T$ is semi-regular for all $|\lambda| < \gamma(T)$, where $\gamma(T)$ denotes the reduced minimal modulus.*

The *semi-regular spectrum* defined as

$$\sigma_{\rm se}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not semi-regular}\}.$$

In literature $\sigma_{\rm se}(T)$ is sometime called the *Kato spectrum* or the *Apostol spectrum*. From Theorem 1.23 we see that $\rho_{\rm se}(T) := \mathbb{C} \setminus \sigma_{\rm se}(T)$ is an open subset of $\mathbb{C}$, so $\sigma_{\rm se}(T)$ is a closed subset of $\mathbb{C}$. Clearly, we have $\sigma_{\rm se}(T) \subseteq \sigma_{\rm ap}(T)$ and $\sigma_{\rm se}(T) \subseteq \sigma_{\rm s}(T)$.

Theorem 1.24. *Let $T \in L(X)$ and consider a connected component Ω of $\rho_{\rm se}(T)$. If $\lambda_0 \in \Omega$ is arbitrarily fixed then*

$$K(\lambda I - T) = K(\lambda_0 I - T) \text{ for every } \lambda \in \Omega.$$

The surjectivity spectrum of an operator is closely related to the local spectra:

Theorem 1.25. *For every operator $T \in L(X)$ we have*

$$\sigma_{\mathrm{su}}(T) = \bigcup_{x \in X} \sigma_T(x).$$

Proof. If $\lambda \notin \bigcup_{x \in X} \sigma_T(x)$ then $\lambda \in \rho_T(x)$ for every $x \in X$ and hence, directly from the definition of $\rho_T(x)$, we conclude that $(\lambda I - T)y = x$ always admits a solution for every $x \in X$, so $\lambda I - T$ is surjective and hence $\lambda \notin \sigma_{\mathrm{su}}(T)$.

Conversely, suppose $\lambda \notin \sigma_{\mathrm{su}}(T)$. Then $\lambda I - T$ is surjective and therefore $X = K(\lambda I - T)$. From Theorem 1.21 it follows that $0 \notin \sigma_{\lambda I - T}(x)$ for every $x \in X$, and consequently $\lambda \notin \sigma_T(x)$ for every $x \in X$. ∎

For an isolated point λ_0 of $\sigma(T)$ the quasi-nilpotent part $H_0(\lambda_0 I - T)$ and the analytical core $K(\lambda_0 I - T)$ may be precisely described as a range or a kernel of the spectral projection P_0 associated with the spectral subset $\{\lambda_0\}$.

We now give a characterization of the isolated points of $\sigma_{\mathrm{s}}(T)$. The following result is taken from [52].

Theorem 1.26. *If $T \in L(X)$ then $X = H_0(\lambda I - T) + K(\lambda I - T)$ if and only if $\sigma_{\mathrm{s}}(T)$ does not cluster at λ.*

Proof. We can take $\lambda = 0$. The equivalence is obvious if $0 \notin \sigma_{\mathrm{s}}(T)$, since $K(\lambda I - T) = X$ in this case. Suppose that $0 \in \sigma_{\mathrm{s}}(T)$. By Theorem 1.12 and Theorem 1.11 we have

$$X = \mathcal{X}_T(\sigma_{\mathrm{s}}(T)) = \mathcal{X}_T(\{0\}) + \mathcal{X}_T(\sigma_{\mathrm{s}}(T) \setminus \{0\}).$$

But, by Theorem 1.21, we have

$$\mathcal{X}_T(\sigma_{\mathrm{s}}(T) \setminus \{0\}) \subseteq \mathcal{X}_T(\mathbb{C} \setminus \{0\}) = K(T),$$

from which we obtain $H_0(T) + K(T) = X$.

Conversely, suppose that $0 \in \sigma_{\mathrm{s}}(T)$ and $H_0(T) + K(T) = X$. Then every $x \in X$ may be written $x = x_1 + x_2$, where $x_1 \in H_0(T)$ and $x_2 \in K(T)$. Clearly, from the definition of $H_0(T)$, we have $\sigma_T(x_1) \subseteq \{0\}$, while $0 \notin \sigma_T(x_2)$, by Theorem 1.11. Therefore,

$$\sigma_T(x) \subseteq \sigma_T(x_1) \cup \sigma_T(x_2) \subseteq \{0\} \cap \sigma_T(x_2),$$

and this implies, since $\sigma_T(x_2)$ is closed, that 0 is isolated in $\sigma_T(x) \subseteq \{0\}$. Since, by Theorem 1.25, there exists $x_0 \in X$ for which $\sigma_T(x_0) = \sigma_{\mathrm{s}}(T)$, we then conclude that 0 is isolated in $\sigma_{\mathrm{s}}(T)$. ∎

The next corollary is an obvious consequence of Theorem 1.26, once observed that equality $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{s}}(T^*)$.

Corollary 1.27. *If $T \in L(X)$ then $X^* = H_0(\lambda I - T^*) + K(\lambda I - T^*)$ if and only if $\sigma_{\mathrm{ap}}(T)$ does not cluster at λ*

Theorem 1.26 has some other interesting consequences:

Corollary 1.28. *If $T \in L(X)$ the following assertions hold:*

(i) *$X = H_0(\lambda I - T) + K(\lambda I - T)$ if and only if $\sigma_T(x)$ does not cluster at λ for every $x \in X$.*

(ii) *$X = H_0(\lambda I - T) + K(\lambda I - T)$ for all $\lambda \in \mathbb{C}$ if and only if $\sigma(T)$ is finite.*

Proof. (i) The direct implication is clear from the proof of Theorem 1.26. For the converse, note that if $\lambda \notin \sigma_{\mathrm{s}}(T)$ then $K(\lambda I - T) = X$. Moreover, $\sigma_T(x) \subseteq \sigma_{\mathrm{s}}(T)$ for all $x \in X$. The converse implication then is a direct consequence of Theorem 1.26.

(ii) Since a compact set consisting of isolated points is a finite set, then Theorem 1.26 entails that $\sigma_{\mathrm{s}}(T)$ is finite, and hence also $\sigma(T)$ is finite. ∎

Remark 1.29. Since the condition $X = H_0(\lambda I - T) + K(\lambda I - T)$ may be thought of as being dual to the condition $H_0(\lambda I - T) \cap K(\lambda I - T) = \{0\}$, one is tempting to conjecture that λ is isolated in $\sigma_{\mathrm{ap}}(T)$ if and only if $H_0(\lambda I - T) \cap K(\lambda I - T) = \{0\}$ and $H_0(\lambda I - T) \cap K(\lambda I - T)$ is closed. The following example shows that this is not true. Recall that the injectivity modulus of $T \in L(X)$ is defined as

$$j(T) := \inf_{\|x\|=1} \|Tx\| = \inf_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

Set

$$i(T) := \lim_{n \to \infty} j(T^n)^{1/n}.$$

Let us consider the *weighted right shift* $S \in L(X)$, where $X = \ell^2(\mathbb{N})$, defined by $Se_n := s_n e_{n+1}$, where (e_n) is the canonical basis of $\ell^2(\mathbb{N})$, and (s_n) is a given weight sequence, with $0 < s_n \leq 1$. We may choose the sequence (s_n) such that $i(S) = 0$ and $r(S) > 0$, $r(S)$ the spectral radius of S, see [68, Chap. 1, §1.6]. By [68, Prop. 1.6.15] we have

$$\sigma_{\mathrm{ap}}(S) = \{\lambda \in \mathbb{C} : i(S) \leq |\lambda| \leq r(S)\},$$

so 0 is not isolated in $\sigma_{\mathrm{ap}}(S)$. Moreover, by [68, Prop. 1.6.16], the subspaces $\mathcal{X}_S(F)$ are closed for all closed $F \subseteq \mathbb{C}$, in particular $H_0(S) = \mathcal{X}_S(\{0\})$ is closed, and

$$K(S) = \bigcap_{n=0}^{\infty} S^n(X) = \{0\}.$$

Theorem 1.30. *Let $T \in L(X)$ and suppose that λ_0 is an isolated point of $\sigma(T)$. If P_0 is the spectral projection associated with $\{\lambda_0\}$, then:*

(i) $P_0(X) = H_0(\lambda_0 I - T)$;

(ii) $\ker P_0 = K(\lambda_0 I - T)$.

Therefore, $X = H_0(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$.

Proof. (i) Since λ_0 is an isolated point of $\sigma(T)$ there exists a positively oriented circle $\Gamma := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| = \delta\}$ which separates λ_0 from the remaining part of the spectrum. We have

$$(\lambda_0 I - T)^n P_0 x = \frac{1}{2\pi i} \int_\Gamma (\lambda_0 - \lambda)^n (\lambda I - T)^{-1} x \, d\lambda \quad \text{for all } n = 0, 1, \cdots .$$

Now, assume that $x \in P_0(X)$. We have $P_0 x = x$ and it is easy to verify the following estimate:

$$\|(\lambda_0 I - T)^n x\| \leq \frac{1}{2\pi} 2\pi \delta^{n+1} \max_{\lambda \in \Gamma} \|(\lambda I - T)^{-1}\| \|x\|.$$

Obviously this estimate also holds for some $\delta_o < \delta$ (since Γ lies in $\rho(T)$), and consequently

$$\limsup \|(\lambda_0 I - T)^n x\|^{1/n} < \delta. \tag{7}$$

This proves the inclusion $P_0(X) \subseteq H_0(\lambda_0 I - T)$.

Conversely, assume that $x \in H_0(\lambda_0 I - T)$ and hence that the inequality (7) holds. Let $S \in L(X)$ denote the operator

$$S := \frac{\lambda_0 I - T}{\lambda_0 - \lambda}.$$

Evidently the Neumann series

$$\sum_{n=0}^{\infty} S^n x = \sum_{n=0}^{\infty} \left(\frac{\lambda_0 I - T}{\lambda_0 - \lambda} \right)^n x$$

converges for all $\lambda \in \Gamma$. If y_λ denotes its sum for every $\lambda \in \Gamma$, from a standard argument of functional analysis we obtain that $(I - S)y_\lambda = x$. A simple calculation also shows that $y_\lambda = (\lambda - \lambda_0)R_\lambda x$ and therefore

$$R_\lambda x = -\sum_{n=0}^{\infty} \frac{\lambda_0 I - T)^n x}{(\lambda_0 - \lambda)^{n+1}} \quad \text{for all } \lambda \in \Gamma.$$

A term by term integration then yields

$$P_0 x = \frac{1}{2\pi i}\int_\Gamma R_\lambda x \, d\lambda = -\frac{1}{2\pi i}\int_\Gamma \frac{1}{(\lambda_0 - \lambda)} x \, d\lambda = x,$$

so $x \in P_0(X)$ and this proves the inclusion $H_0(\lambda_0 I - T) \subseteq P_0(X)$, so the proof of (i) is complete.

(ii) There is no harm in assuming that $\lambda_0 = 0$. We have $\sigma(T|P_0(X)) = \{0\}$, and $0 \in \rho(T|\ker P_0)$. From the equality $T(\ker P_0) = \ker P_0$ we obtain $\ker P_0 \subseteq K(T)$, see Theorem 1.20. It remains to prove the reverse inclusion $K(T) \subseteq \ker P_0$. To see this we first show that $H_0(T) \cap K(T) = \{0\}$. This is clear because $H_0(T) \cap K(T) = K(T|H_0(T))$, and the last subspace is $\{0\}$ since the restriction of T on the Banach space $H_0(T)$ is a quasi-nilpotent operator, see Corollary 3.7. Hence $H_0(T) \cap K(T) = \{0\}$. From this it then follows that

$$\begin{aligned} K(T) &\subseteq K(T) \cap X = K(T) \cap [\ker P_0 \oplus P_0(X)] \\ &= \ker P_0 + K(T) \cap H_0(T) = \ker P_0, \end{aligned}$$

so the desired inclusion is proved. ∎

If λ_0 is a pole of the resolvent we can say much more:

Corollary 1.31. *Let $T \in L(X)$ and suppose that λ_0 is a pole of the resolvent of T, or equivalently $p := p(\lambda I - T) = q(\lambda I - T) < \infty$. Then*

$$H_0(\lambda_0 I - T) = \ker(\lambda_0 I - T)^p,$$

and

$$K(\lambda_0 I - T) = (\lambda_0 I - T)^p(X).$$

We now consider the isolated points of $\sigma_{\mathrm{ap}}(T)$. The following two results are from [52].

Theorem 1.32. *Suppose that $T \in L(X)$ and λ an isolated point of $\sigma_{\mathrm{ap}}(T)$. Then*

(i) *Both $H_0(\lambda I - T)$ and $K(\lambda I - T)$ are closed subspaces.*

(ii) $H_0(\lambda I - T) \cap K(\lambda I - T) = \{0\}$.

(iii) *The direct sum $H_0(\lambda I - T) \oplus K(\lambda I - T)$ is closed and there exists $\lambda_0 \neq 0$ such that*

$$H_0(\lambda I - T) \oplus K(\lambda I - T) = K(\lambda_0 I - T) = \bigcup_{n=0}^{\infty} T(\lambda_0 I - T)^n(X).$$

Proof. We may assume $\lambda = 0$. Since 0 an isolated point of $\sigma_{\mathrm{ap}}(T)$, there exists a $\delta > 0$ such that $\lambda I - T$ is bounded below for all $0 < |\lambda| < \delta$. By Theorem 1.24, the map $\lambda \to K(\lambda I - T)$ is constant on the punctured disc $\mathbb{D}(0, \delta) \setminus \{0\}$, and fixing $\lambda_0 \in \mathbb{D}(0, \delta) \setminus \{0\}$ we have, by Theorem 1.23, that $K(\lambda_0 I - T) = (\lambda_0 I - T)^\infty(X)$. Set $X_0 := (\lambda_0 I - T)^\infty(X)$, and denote by $T_0 : X_0 \to X_0$ the operator induced by T on the Banach space X_0. Clearly, $\lambda I - T_0$ is bijective for all $\lambda \in \mathbb{D}(0, \delta) \setminus \{0\}$. Since T is not onto then $K(T) \neq X$, hence $H_0(T) \neq 0$ by Theorem 1.26. By Theorem 1.11, part (ii), we know that

$$(\lambda I - T)(H_0(T)) = (\lambda I - T)(\mathcal{X}_T(\{0\}) = \mathcal{X}_T(\{0\} = H_0(T) \quad \text{for all } \lambda \neq 0,$$

from which we deduce that

$$(\lambda I - T)^n(H_0(T)) = H_0(T) \subseteq (\lambda I - T)^n(X) \quad \text{for all } n \in \mathbb{N},$$

so that $H_0(T) \subseteq X_0$, hence $0 \in \sigma(T_0)$, and since $\lambda I - T_0$ is bijective for all $\lambda \in \mathbb{D}(0, \delta) \setminus \{0\}$, we then have that 0 is an isolated point of $\sigma(T_0)$. By Theorem 1.30, $X_0 = H_0(T_0) \oplus K(T_0)$. Clearly, $H_0(T) = H_0(T_0)$, so to finish the proof it suffices to prove $K(T) = K(T_0)$. Let $x_0 \in K(T)$, $Tx_{n+1} = x_n$ and $\|x_n\| \leq c^n \|x_0\|$ for all n. Then $\phi(\lambda) := \sum_{n=0}^{\infty} x_{n+1}\lambda^n$ defines an analytic function on the open disc $\mathbb{D}(0, \frac{1}{c})$ that satisfies

$$(\lambda I - T)(\phi(\lambda) = \sum_{n=0}^{\infty} x_n \lambda^n - \sum_{n=0}^{\infty} x_{n+1}\lambda^{n+1} \quad \text{for all } \lambda \in \mathbb{D}(0, \frac{1}{c}).$$

In particular, $x_0 \in (\lambda I - T)(X)$ for all $\lambda \in \mathbb{D}(0, \frac{1}{c})$. Therefore, $x_0 \in K(\lambda_0 I - T)$, and hence $K(T) \subseteq X_0$. Note that

$$\phi(\lambda) = (\lambda I - T_0)^{-1} x_0 \in X_0 \quad \text{for all } \lambda \in \mathbb{D}(0, \frac{1}{c}).$$

By continuity, $x_1 = \phi(0) \in X_0$. A similar argument shows that $x_n \in X_0$ for $n \geq 1$, thus $x_0 \in K(T_0)$, from which we conclude that $K(T) = K(T_0)$. ∎

Corollary 1.33. *Let $T \in L(X)$ and suppose that $\dim K(T) < \infty$. Then T has SVEP.*

Proof. We know that $\ker(\lambda I - T) \subseteq K(T)$ for each $\lambda \neq 0$, hence $\ker(\lambda I - T)$ is finite -dimensional. Moreover, a set of eigenvectors, each of them corresponding to a different eigenvalue of T, is linearly independent, so our assumption $\dim K(T) < \infty$ implies that the punctual spectrum $\sigma_{\rm p}(T)$ is finite, and consequently T has SVEP. ∎

The condition $\dim K(T) < \infty$ is clearly satisfied if $T^\infty(X) = \{0\}$, since $K(T) \subseteq T^\infty(X)$. The condition $T^\infty(X) = \{0\}$ may thought as an abstract shift condition since is satisfied by every unilateral weighted right shift, see [1, Chapter 2, §5].

It has been observed before, the local spectral subspaces $X_T(\Omega)$ need not be closed, also in the case that T has the SVEP. In fact, the operator T of Example 1.17 has the SVEP, its quasi-nilpotent part $H_0(T)$ is not closed and, by Theorem 1.14, $H_0(T) = X_T(\{0\}$. On the other hand, for every spectral operator T with spectral measure $E(\cdot)$, the subspace $X_T(\Omega)$ is closed, for every closed set F, since it coincides with the range of the projection $E(F)$, see [68, Corollary 1.2.25].

The following property dates back to the earliest days of local spectral theory, and was introduced first by Dunford (see [46] and plays an important role in the development of the theory of spectral operators and more in general in the development of decomposable operators.

Definition 1.34. *A bounded operator $T \in L(X)$, X a Banach space, is said to have the Dunford's property (C), shortly the property* (C)*, if the analytic subspace $X_T(\Omega)$ is closed for every closed subset $\Omega \subseteq \mathbb{C}$.*

Trivially, by Theorem 1.9, we have the following relevant fact:

Theorem 1.35. *If $T \in L(X)$, X a Banach space, has the property* (C) *then T has the SVEP.*

Note that if an operator T has the property (C), and hence the SVEP, then the quasi-nilpotent part $H_0(T)$ is closed since $H_0(T) = X_T(\{0\})$, see Theorem 1.14. The operator T considered in Example 1.17 shows that the implication of Theorem 1.35 cannot be reversed in general. Further examples of operators with the SVEP but without the property (C) may be found among the class of all multipliers of semi-simple commutative Banach algebras. These operators have the SVEP, since the quasi-nilpotent part of $\lambda I - T$ coincides the kernel $\ker(\lambda I - T)$ for all $\lambda \in \mathbb{C}$, see [1, Theorem 4.33], while the property (C) plays a distinctive role in this context, see [68, Chapter 4].

A first example of operators which have the property (C) is given by quasi-nilpotent operators.

Theorem 1.36. *Let $T \in L(X)$ be a quasi-nilpotent operator on a Banach space X. Then T has the property* (C).

Proof. Consider any closed subset $F \subseteq \mathbb{C}$. Consider first the case $0 \notin F$. Then since T has the SVEP

$$X_T(F) = X_T(F \cap \sigma(T)) = X_T(\emptyset) = \{0\}$$

is trivially closed. On the other hand, if $0 \in F$ then, by Theorem 1.18 and Theorem 1.14, we have

$$X_T(F) = X_T(F \cap \sigma(T)) = X_T(\{0\}) = H_0(T) = X.$$

Hence, also in this case $X_T(F)$ is closed. ∎

The property (C) is inherited by restrictions to closed invariant subspaces. Note that if T has the property (C) then so does $f(T)$ for every function f analytic on an open neighborhood $\mathcal{U}$ of $\sigma(T)$, see Theorem 3.3.6 of Laursen and Neumann [68]. It could be reasonable to expect that the converse is true if we assume that f is non-constant on each connected component of $\mathcal{U}$, as it happens, by Theorem 3.15, for the SVEP; but this is not known.

Let X and Y be Banach spaces and consider two operators $S \in L(X,Y)$ and $R \in L(Y,X)$. The common spectral properties of the operators RS and SR have been studied by a number of authors. It is known that the non-zero points of the spectrum of the products RS and SR are the same, the same holds for a number of its more distinguished parts, see for instance [22]. We have shown in Theorem 1.2 that the local spectrum RS at x and the local spectrum of SR at Sx have the same non-zero points. In the remaining part of this section we shall consider some other local spectral properties of SR and RS.

Lemma 1.37. *Let $S \in L(X,Y)$ and $R \in L(Y,X)$. Then SR has SVEP at λ if and only if RS has SVEP at λ.*

Proof. The proof is rather easy. ∎

In the final part of this section we give some results from [10] and [98].

Theorem 1.38. *Let F be a closed subset of $\mathbb{C}$ such that $0 \in F$. If $S \in L(X,Y)$ and $R \in L(Y,X)$ then $Y_{SR}(F)$ is closed if and only if $X_{RS}(F)$ is closed.*

Proof. Suppose that $Y_{SR}(F)$ is closed and let (x_n) be a sequence in $X_{RS}(F)$ which converges to $x \in X$. Since $x_n \in X_{RS}(F)$ then $\sigma_{RS}(x_n) \subseteq F$ for all $n \in \mathbb{N}$. Since $0 \in F$ then $\sigma_{RS}(x_n) \cup \{0\} \subseteq F$. By Theorem 1.2, part (i), we have $\sigma_{RS}(x_n) \cup \{0\} = \sigma_{SR}(Sx_n) \cup \{0\}$, so $\sigma_{SR}(Sx_n) \subseteq F$ and hence $Sx_n \in Y_{SR}(F)$. But $Sx_n \to Sx$ and $Y_{SR}(F)$ is closed, thus $Sx \in Y_{SR}(F)$, that is $\sigma_{SR}(Sx) \subseteq F$. Again by Theorem 1.2 we have $\sigma_{RS}(x) \subseteq \sigma_{SR}(Sx) \cup \{0\} \subseteq F$, thus $x \in X_{RS}(F)$.

The converse implication follows in a similar way, just use part (ii) of Theorem 1.2. ∎

In order to study the case that $0 \notin F$ we need two preliminary results:

Lemma 1.39. *Suppose that $T \in L(X)$ has SVEP, and that $F \subseteq \mathbb{C}$ is a closed set for which $X_T(F)$ is closed. Then $\sigma(T|X_T(F)) \subseteq F \cap \sigma(T)$.*

Proof. Set $A := T|X_T(F)$. Clearly, $\lambda I - A$ has SVEP and part (ii) of Theorem 1.8 ensures that $\lambda I - S$ is onto for all $\lambda \in \mathbb{C} \setminus F$. By Corollary 3.2 then $\lambda I - S$ is invertible for all $\lambda \in \mathbb{C} \setminus F$. on the other hand, part (iii) of Theorem 1.8 shows that $\lambda I - S$ is invertible for all $\lambda \in F$ which belongs to the resolvent, so $\sigma(S) \subseteq (\mathbb{C} \setminus F) \cup \sigma(T)$, and hence $\sigma(S) \subseteq F \cap \sigma(T)$. ∎

Lemma 1.40. *Suppose that $T \in L(X)$ has SVEP and let F be a closed subset of $\mathbb{C}$ such that $Z := X_T(F)$ is closed. If $A := T|X_T(F)$ then $X_T(K) = Z_A(K)$ for all closed $K \subseteq F$.*

Proof. Note first that A has SVEP, so the glocal subspace $\mathcal{Z}_A(K)$ coincides with the local subspace $Z_A(K)$, and $X_T(K) \subseteq X_T(F) = Z$. The inclusion $\mathcal{Z}_A(K) \subseteq \mathcal{X}_T(K)$ is immediate. In order to prove the opposite inclusion, suppose that $x \in X_T(K) = \mathcal{X}_T(K)$. Then $\sigma_T(x) \subseteq K$ and there is an analytic function $f : \mathbb{C} \setminus K \to X$ such that $(\mu I - T)f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus K$. We also have

$$\sigma_T(f(\mu)) = \sigma_T(x) \subseteq K \quad \text{for all } \mu \in \mathbb{C} \setminus K,$$

thus $f(\mu) \in X_T(K) \subseteq Z$. Therefore, f is a Z-valued function and hence

$$(\mu I - T)f(\mu) = (\mu I - A)f(\mu) = x \quad \text{for all } \mu \in \mathbb{C} \setminus K,$$

i.e. $x \in \mathcal{Z}_A(K) = Z_A(K)$. ∎

Theorem 1.41. *Let F be a closed subset of $\mathbb{C}$ such that $\lambda \notin F$. If $T \in L(X)$ has SVEP and $X_T(F \cup \{\lambda\})$ is closed then $X_T(F)$ is closed.*

Proof. Let $Z := X_T(F \cup \{\lambda\})$ and $S := T|X_T(F \cup \{\lambda\})$. From Lemma 1.39 we know that $\sigma(S) \subseteq F \cup \{\lambda\}$. We consider two cases: Suppose first that $\lambda \notin \sigma(S)$. Then $\sigma(S) \subseteq F$ and hence $Z = Z_S(F)$. By Lemma 1.40 we then have $Z_S(F) = X_T(F)$, so $X_T(F)$ is closed. Suppose the other case that $\lambda \in \sigma(S)$ and set $F_0 := \sigma(S) \cap F$. Then $\sigma(S) = F_0 \cup \{\lambda\}$. Since $\lambda \in \sigma(S)$, from part (vi) of Theorem 1.11 we have $Z = Z_S(F_0) \oplus Z_S(\{\lambda\}$. From Lemma 1.40 it then follows

$$Z_S(F_0) = Z_F(\sigma(S) \cap F) = Z_S(F) = X_T(F),$$

and hence $X_T(F)$ is closed. ∎

Corollary 1.42. *Let $S \in L(X, Y)$ and $R \in L(Y, X)$ be such that RS has SVEP, and denote by F a closed subset of $\mathbb{C}$ such that $0 \notin F$. Then we have:*

(i) *If $Y_{SR}(F \cup \{0\})$ is closed then $X_{RS}(F)$ is closed.*

(ii) *If $X_{RS}(F \cup \{0\})$ is closed then $Y_{SR}(F)$ is closed.*

Proof. Theorem 1.38 ensures that $Z := X_{RS}(F \cup \{0\})$ is closed, since $0 \in F \cup \{0\}$. The SVEP for RS entails the SVEP for SR, thus by Theorem 1.41 we deduce that $X_{RS}(F)$ is closed. An analogous argument proves (ii). ∎

Corollary 1.43. *If $S \in L(X, Y)$ and $R \in L(Y, X)$ then RS has Dunford's property (C) if and only if SR has Dunford's property (C).*

Proof. Suppose that RS has Dunford's property (C), i.e. $X_{RS}(F)$ is closed for every closed subset $F \subseteq \mathbb{C}$. If $0 \in F$ then $Y_{SR}(F)$ is closed, by Theorem 1.38. Obviously, if $0 \notin F$ then $F \cup \{0\}$ is closed, so $X_{RS}(F \cup \{0\})$ is closed and hence $Y_{SR}(F)$ is closed, by Corollary 1.42. Therefore SR has Dunford's property (C). The proof of the opposite implication is similar. ∎

2. Classes of operators in Fredholm theory

This section concerns many classes of operators which arise from the classical Fredholm theory of bounded operators on Banach spaces. The first part addresses with some preliminary and basic notions, concerning some important invariant subspaces, the hyper-range, the hyper-kernel, and the analytic core of an operator. The importance of the role of these subspaces becomes evident when we will study the special classes of operators treated in the second part.

Given a linear bounded operator $T \in L(X, Y)$, the *kernel* of T is denoted by $\ker T$, while the *range* of T is denoted by $T(X)$. In the sequel, for every bounded operator $T \in L(X, Y)$, we shall denote by $\alpha(T)$ the *nullity* of T, defined as $\alpha(T) := \dim \ker T$, while the *deficiency* $\beta(T)$ of T is defined $\beta(T) := \operatorname{codim} T(X) = \dim Y/T(X)$.

The kernels and the ranges of the iterates T^n, $n \in \mathbb{N}$, of a linear operator T defined on a vector space X, form two increasing and decreasing chains, respectively, i.e. the chain of kernels

$$\ker T^0 = \{0\} \subseteq \ker T \subseteq \ker T^2 \subseteq \cdots$$

and the chain of ranges

$$T^0(X) = X \supseteq T(X) \supseteq T^2(X) \cdots .$$

The subspace

$$\mathcal{N}^\infty(T) := \bigcup_{n=1}^{\infty} \ker T^n$$

is called the *hyper-kernel* of T, while

$$T^\infty(X) := \bigcap_{n=1}^{\infty} T^n(X)$$

is called the *hyper-range* of T. Note that both $\mathcal{N}^\infty(T)$ and $T^\infty(X)$ are T-invariant linear subspace of T, i.e.

$$T(\mathcal{N}^\infty(T)) \subseteq \mathcal{N}^\infty(T) \quad \text{and} \quad T(T^\infty(X)) \subseteq T^\infty(X).$$

In the next result we give some useful connections between the kernels and the ranges of the iterates T^n of an operator T on a vector space X.

Theorem 2.1. *For a linear operator T on a vector space X the following statements are equivalent:*

(i) $\ker T \subseteq T^m(X)$ *for each* $m \in \mathbb{N}$*;*

(ii) $\ker T^n \subseteq T(X)$ *for each* $n \in \mathbb{N}$*;*

(iii) $\ker T^n \subset T^m(X)$ *for each* $n \in \mathbb{N}$ *and each* $m \in \mathbb{N}$*;*

(iv) $\ker T^n = T^m(\ker T^{m+n})$ *for each* $n \in \mathbb{N}$ *and each* $m \in \mathbb{N}$*.*

(v) $\ker T \subseteq T^\infty(X)$*;*

(vi) $\mathcal{N}^\infty(T) \subseteq T(X)$*;*

(vii) $\mathcal{N}^\infty(T) \subseteq T^\infty(X)$*.*

From Theorem 2.1 we then see that $T \in L(X)$ is semi-regular if $T(X)$ is closed and one of the equivalent conditions (i)-(vii) holds.

The class of all *upper semi-Fredholm* operators defined by

$$\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\}$$

and the class of all *lower semi-Fredholm* operators, defined by

$$\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}$$

The class of all *semi-Fredholm operators* is defined as $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, while the class of the *Fredholm* operators is defined as $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$. The *index* of $T \in \Phi_\pm(X)$ is defined by $\text{ind}\,(T) := \alpha(T) - \beta(T)$. The *essential spectrum* (or the Fredholm spectrum) is defined by

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi(X)\}.$$

The essential spectrum is a non-empty closed subset of $\mathbb{C}$. The classical punctured neighborhood theorem, see [78, Chap. III, Theorem 7], for semi-Fredholm operators, says that if $T \in \Phi_+(X)$ then there exists $\varepsilon > 0$ such that $\lambda I + T \in \Phi_+(X)$ and $\alpha(\lambda I + T)$ is constant on the punctured neighbourhood $0 < |\lambda| < \varepsilon$. Moreover,

$$\alpha(\lambda I + T) \leq \alpha(T) \quad \text{for all } |\lambda| < \varepsilon\,, \tag{8}$$

and

$$\text{ind}\,(\lambda I + T) = \text{ind}\, T \quad \text{for all } |\lambda| < \varepsilon.$$

Analogously, if $T \in \Phi_-(X)$ then there exists $\varepsilon > 0$ such that $\lambda I + T \in \Phi_-(X)$, $\beta(\lambda I + T)$ is constant on the punctured neighbourhood $0 < |\lambda| < \varepsilon$. Moreover,

$$\beta(\lambda I + T) \leq \beta(T) \quad \text{for all } |\lambda| < \varepsilon, \tag{9}$$

and

$$\text{ind}\,(\lambda I + T) = \text{ind}\,T \quad \text{for all } |\lambda| < \varepsilon.$$

Definition 2.2. *Let $T \in \Phi_\pm(X)$, X a Banach space. Let $\varepsilon > 0$ as in (8) or (9). If $T \in \Phi_+(X)$ the* jump $j(T)$ *is defined by*

$$j(T) := \alpha(T) - \alpha(\lambda I + T), \quad 0 < |\lambda| < \varepsilon,$$

while, if $T \in \Phi_-(X)$, the jump $j(T)$ is defined by

$$j(T) := \beta(T) - \beta(\lambda I + T), \quad 0 < |\lambda| < \varepsilon.$$

The continuity of the index ensures that both definitions of the jump coincide whenever T is a Fredholm operator. In general, semi-Fredholm operators are not semi-regular. In fact we have (see for a proof [1, Theorem 1.58]):

Theorem 2.3. *[1, Theorem 1.58] A semi-Fredholm operator $T \in L(X)$ is semi-regular precisely when $j(T) = 0$.*

The following definition from was originated by Kato's classical treatment [63] of perturbation theory of semi-Fredholm operators.

Definition 2.4. *An operator $T \in L(X)$, X a Banach space, is said to admit a* generalized Kato decomposition, *abbreviated as GKD, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T|M$ is semi-regular and $T|N$ is quasi-nilpotent.*

Evidently, every semi-regular operator has a GKD $M = X$ and $N = \{0\}$ and a quasi-nilpotent operator has a GKD $M = \{0\}$ and $N = X$.

A relevant case is obtained if we assume in the definition above that $T|N$ is nilpotent, i.e. there exists $d \in \mathbb{N}$ for which $(T|N)^d = 0$. In this case T is said to be of *Kato type of operator of order d*.

An operator $T \in L(X)$ is said to be *essentially semi-regular* if it admits a GKD (M, N) such that N is finite-dimensional. Note that if T is essentially semi-regular then $T|N$ is nilpotent, since every quasi-nilpotent operator on a finite dimensional space is nilpotent.

Hence we have the following implications:

$$\begin{aligned} T \text{ semi-regular} \quad &\Rightarrow \quad T \text{ essentially semi-regular } \Rightarrow T \text{ of Kato type} \\ &\Rightarrow \quad T \text{ admits a GKD.} \end{aligned}$$

In the sequel we reassume some results concerning essentially semi-regular operators. The reader may be found a well-organized exposition of the basic results concerning this class of operators in Müller [78, §21], where essentially semi-regular operators are called *essentially Kato operators*.

(i) $T \in L(X)$ is essentially semi-regular if and only if $T(X)$ is closed an there exists a finite-dimensional subspace F of X such that $\ker T \subseteq T^\infty(X) + F$.

(ii) If $T \in L(X)$ is essentially semi-regular then T^n is essentially semi-regular for every $n \in \mathbb{N}$.

(iii) $T \in L(X)$ is essentially semi-regular if and only if $T^* \in L(X^*)$ is essentially semi-regular.

(iii) If T and S commutes and TS is essentially semi-regular then both T and S are essentially semi-regular.

(iv) If $T \in L(X)$ is essentially semi-regular there exists $\varepsilon > 0$ such that $T + S$ is essentially semi-regular for every $S \in L(X)$ such that $ST = TS$ and $\|S\| < \varepsilon$.

(v) If $T \in L(X)$ is essentially semi-regular then $T + K$ is essentially semi-regular for every finite-rank operator $K \in L(X)$.

By Theorem 2.3 we know that semi-Fredholm operators in general are not semi-regular. The following important result was first observed by Kato [63] , for a proof see [76, Theorem 16.21].

Theorem 2.5. *Every semi-Fredholm operator $T \in L(X)$ is essentially semi-regular, in particular is of Kato type.*

We now introduce two important notions in operator theory.

Definition 2.6. *Given a linear operator T on a vector space X, T is said to have* finite ascent *if $\mathcal{N}^\infty(T) = \ker T^k$ for some positive integer k. Clearly, in such a case there is a smallest positive integer $p = p(T)$ such that $\ker T^p = \ker T^{p+1}$. The positive integer p is called the* ascent *of T. If there is no such integer we set $p(T) := \infty$. Analogously, T is said to have* finite descent *if $T^\infty(X) = T^k(X)$ for some k. The smallest integer $q = q(T)$ such that $T^{q+1}(X) = T^q(X)$ is called the* descent *of T. If there is no such integer we set $q(T) := \infty$.*

Some important basic properties are reassumed in the following theorem:

Theorem 2.7. *[1, Theorem 3.4] If T is a linear operator on a vector space X then the following properties hold:*

(i) *$p(T) \le m < \infty$ if and only if for every natural n we have $T^m(X) \cap \ker T^n = \{0\}$.*

(ii) *$q(T) \le m < \infty$ if and only if for every $n \in \mathbb{N}$ there exists a subspace $Y_n \subseteq \ker T^m$ such that $X = Y_n \oplus T^n(X)$.*

(iii) *If $p(T)$ and $q(T)$ are finite, then $p(T) = q(T)$.*

(iv) *If $p(T) < \infty$ then $\alpha(T) \le \beta(T)$.*

(v) *If $q(T) < \infty$ then $\beta(T) \le \alpha(T)$.*

(vi) *If $p(T) = q(T) < \infty$ then $\alpha(T) = \beta(T)$ (possibly infinite).*

(vii) *If $\alpha(T) = \beta(T) < \infty$ and if either $p(T)$ or $q(T)$ is finite then $p(T) = q(T)$.*

Some special classes of semi-Fredholm operators are given by the class $B_+(X)$ of all *upper semi-Browder operators*, defined as

$$B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\},$$

and by the class the class $B_-(X)$ of all *lower semi-Browder operators*, defined as

$$B_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}.$$

The class of all *Browder operators* is defined by $B(X) = B_+(X) \cap B_-(X)$. Clearly,

$$B_+(X) := \{T \in \Phi(X) : p(T) = q(T) < \infty\}.$$

Theorem 2.8. *[78, §21] For a bounded operator $T \in L(X)$, the following conditions are equivalent:*

(i) *T is essentially semi-regular;*

(ii) *there exists a closed T-invariant subspace Y of X such that the restriction $T|Y$ is lower semi-Fredholm and the induced operator $\tilde{T} : X/Y \to X/Y$ is upper semi-Fredholm;*

(iii) *there exists a closed T-invariant subspace Y of X such that the restriction $T|Y$ is lower semi-Browder and the induced operator $\tilde{T} : X/Y \to X/Y$ is upper semi-Browder;*

(iv) *there exists a closed T-invariant subspace Y of X such that the restriction $T|Y$ is onto and the induced operator $\tilde{T} : X/Y \to X/Y$ is upper semi-Browder;*

(v) *there exists a closed T-invariant subspace Y of X such that the restriction $T|Y$ is lower semi-Browder and the induced operator $\tilde{T} : X/Y \to X/Y$ is bounded below.*

In the sequel by ∂K we denote the boundary of $K \subseteq \mathbb{C}$.

Theorem 2.9. *[1, Theorem 1.75]Let $T \in L(X)$, $X \neq \{0\}$ a Banach space. Then semi-regular spectrum $\sigma_{\rm se}(T)$ is a non-empty compact subset of $\mathbb{C}$ containing $\partial\sigma(T)$. In particular, $\partial\sigma(T)$ is contained in $\sigma_{\rm ap}(T) \cap \sigma_{\rm s}(T)$.*

Since for every n we have $\ker T^n \subseteq \ker T^{n+1}$ we can consider for every n the mapping

$$\Phi_n : \ker T^{n+2}/\ker T^{n+1} \to \ker T^{n+1}/\ker T^n,$$

induced by T, and defined as

$$\Phi_n(z + \ker T^{n+1}) := Tz + \ker T^n \quad z \in \ker T^{n+2}.$$

Analogously, since $T^{n+1}(X) \subseteq T^n(X)$, we can consider for every n the sequence of mappings

$$\Psi_n : T^n(X)/T^{n+1}(X) \to T^{n+1}(X)/T^{n+2}(X),$$

defined as

$$\Psi_n(z + T^{n+1}(X)) := Tz + T^{n+2}(X), \quad z \in T^n(X).$$

Most of following results concerning the uniform descent may be found in Grabiner [54].

Theorem 2.10. *Let T be a linear operator on a vector space X, and n a nonnegative integer.*

(i) *Every map Ψ_n, induced by T, is onto. Moreover, the kernel of Ψ_n is naturally isomorphic to the quotient* $(\ker T \cap T^n(X))/(\ker T \cap T^{n+1}(X))$.

(ii) *Every map Φ_n, induced by T, is injective. Moreover, the cokernel of Φ_n is naturally isomorphic to to the quotient* $(\ker T^{n+1} + T(X))/(\ker T^n + T(X))$.

(iii) *T^n induces a linear isomorphism from the cokernel of Φ_n onto the kernel of Ψ_n.*

Corollary 2.11. *$k_n(T)$ is equal to the codimension of the image of the linear mapping Ψ_n. Moreover,*

$$k_n(T) = dim\,\frac{\ker T \cap T^n(X)}{\ker T \cap T^{n+1}(X)} = dim\,\frac{\ker T^{n+1} + T(X)}{\ker T^n + T(X)}. \tag{10}$$

Definition 2.12. *A linear operator T on a vector space X and let d be a nonnegative integer. T is said to have* uniform descent for $n \geq d$ *if $k_n(T) = 0$ for all $n \geq d$.*

Theorem 2.13. *Suppose that T is a linear operator on a vector space X.*

(i) *If T has finite nullity $\alpha(T)$, or finite defect $\beta(T)$, then T has uniform descent for $n \geq 0$.*

(ii) *If T has finite descent p then T has uniform descent for $n \geq p$.*

(iii) *If T has finite descent q then T has uniform descent for $n \geq q$.*

Theorem 2.14. *If T is a linear operator on a vector space X and d is a fixed nonnegative integer, then the following statements are equivalent:*

(i) *T has uniform descent for each $n \geq d$;*

(ii) *The sequence of subspaces* $\{\ker T \cap T^n(X)\}$ *is constant for $n \geq d$;*

(iii) $\ker T \cap T^d(X) = \ker T \cap T^\infty(X)$;

(iv) *The maps induced by T from* $\ker T^{n+2}/\ker T^{n+1}$ *to* $\ker T^{n+1}/\ker T^n$ *are isomorphisms for $n \geq d$;*

(v) *The sequence of subspaces* $\{\ker T^n + T(X)\}$ *is constant for $n \geq d$;*

(vi) $\ker T^d + T(X) = \mathcal{N}^\infty(T) + T(X)$.

Let $T \in L(X)$, X a Banach space. The *operator range topology* on $T(X)$ is the topology induced by the norm $\|\cdot\|_T$ defined:

$$\|y\|_T := \inf_{x \in X}\{\|x\| : y = Tx\}.$$

For a detailed discussion of operator ranges and their topology we refer the reader to [47] and [53].

Definition 2.15. *An operator $T \in L(X)$, X a Banach space, is said to have* topological uniform descent for $n \geq d$ *if T has uniform descent for $n \geq d$ and $T^n(X)$ is closed in the operator range topology of $T^d(X)$ for each $n \geq d$.*

The topological uniform descent for $n \geq d$ may be characterized in several ways:

Theorem 2.16. *[54] If $T \in L(X)$, X a Banach space, has uniform descent for $n \geq d$, then the following assertions are equivalent:*

(i) *T has topological uniform descent for $n \geq d$;*

(ii) *There is an integer $n \geq d$ and $k \in \mathbb{N}$ such that $T^{n+k}(X)$ is closed in the operator range topology on $T^n(X)$;*

(iii) *For each $n \geq d$ and $k \in \mathbb{N}$, $T^{n+k}(X)$ is closed in the operator range topology on $T^n(X)$;*

(iv) *There is $n \geq d$ and $k \in \mathbb{N}$, such that* $\ker T^n + T^k(X)$ *is closed in X;*

(v) *For all $n \geq d$ and for all $k \in \mathbb{N}$,* $\ker T^n + T^k(X)$ *(and also for $k = \infty$), is closed in X.*

Some other important properties for operators having topological uniform descent are given in the following theorem.

Theorem 2.17. *[54] Let $T \in L(X)$ be with topological uniform descent for $n \geq d$. Then we have:*

(i) *The restriction of T to $T^\infty(X)$ is onto.*

(ii) *The map induced by T on $T^d(X)/T^\infty(X)$ is bounded below.*

(iii) *The restriction of T to $T^d(X) \cap \overline{\mathcal{N}^\infty(T)}$ is onto.*

(iv) *The map $\hat{T} : X/\overline{\mathcal{N}^\infty(T)} \to X/\overline{\mathcal{N}^\infty(T)}$, defined by $\hat{T}[x] = [Tx]$ is bounded below.*

Essentially semi-regular operators has topological uniform descent. In the remaining part of this section we give some perturbation results of operators T having topological uniform descent. We consider bounded operators S which commute with T for which $T - S$ is "sufficiently small" in the sense of the following definition.

Definition 2.18. *Suppose that $T \in L(X)$, X a Banach space, has topological uniform ascent for $n \geq d$, and let $S \in L(X)$ be an operator which commutes with T. We say that $S - T$ is* sufficiently small *if the norm of restriction $(S - T)|T^d(X)$ is less than the reduced minimun modulus $\gamma(T|T^d(X))$.*

Note that if $T^d(X)$ is closed in X and is given the restriction norm, it is easily seen that $\|S - T\|$ is no greater than the norm of its restriction to $T^d(X)$, so the definition above is essentially a restriction of $\|S - T\|$.

We now consider, the important case is when $S - T$ is invertible. This case, of course, subsumes the case $S = \lambda I - T$, when $\lambda \neq 0$.

Theorem 2.19. *[54] Suppose that $T \in L(X)$ has topological uniform descent for $n \geq d$, and that $S \in L(X)$ commutes with T. If $S - T$ is sufficiently small and is invertible, then:*

(i) *S has closed range and topological uniform descent for $n \geq 0$.*

(ii) *$dim\,((S^n(X))/S^{n+1}(X) = dim\,((T^d(X)/T^{d+1}(X)$, for all $n \geq 0$.*

(iii) *dim* $(\ker (T^{n+1}/ \ker T^n) = dim(\ker (T^{d+1}/ \ker T^d)$ *for all $n \geq 0$.*

(iv) *$S^\infty(X) = T^\infty(X) + \mathcal{N}^\infty(T)$.*

(v) *$\overline{\mathcal{N}^\infty(S)} = \overline{[T^\infty(X) \cap \mathcal{N}^\infty(T)]}$.*

Theorem 2.19 has some important consequences, if we assume further properties on $\ker T \cap T^d(X)$ and $T(X) + \ker T^d$.

Corollary 2.20. *[54] Suppose that $T \in L(X)$ has topological uniform descent for $n \geq d$, and that $S \in L(X)$ commutes with T. If $S - T$ is sufficiently small and is invertible, then the following assertion holds:*

(i) *If* $\ker T \cap T^d(X)$ *has finite dimension, then S is upper semi-Fredholm and $\alpha(S) = dim\,(\ker T \cap T^d(X))$.*

(ii) *If $T(X) + \ker T^d$ has finite codimension then S is lower semi-Fredholm and $\beta(S) = codim\,(T(X) + \ker T^d)$.*

Theorem 2.21. *[54] Suppose that $T \in L(X)$ has topological uniform descent for $n \geq d$, and that $S \in L(X)$ commutes with T. If $S - T$ is sufficiently small and is invertible, then:*

(i) *S has infinite ascent or descent if and only if T does.*

(ii) *S cannot have finite ascent $p(T) > 0$, or finite descent $q(T) > 0$.*

(iii) *S is onto if and only if T has finite descent.*

(iv) *S is injective (or also bounded below) if and only if T has finite descent.*

(v) *S is invertible if and only if $p(T) = q(T) < \infty$.*

The following corollary is just a special case of Theorem 2.21, part (v).

Corollary 2.22. *Suppose that λ belongs to the boundary of the spectrum $\partial\sigma(T)$, and $\lambda I - T$ has uniform descent. Then $p(\lambda I - T) = q(\lambda I - T) < \infty$.*

We conclude this section by proving that the dual of an operator having topological uniform descent may have not topological uniform descent. First we need the following two lemmas.

Lemma 2.23. *Suppose that $T, S \in L(X)$, X a Banach space, satisfy $T(X) \cap S(X) = \{0\}$ and the sum $T(X) \cap S(X)$ is closed. Then $T(X)$ and $S(X)$ are closed.*

Proof. Define $U : X \times X \to X$ by means of $U(x, y) := Tx + Sy$ for all $(x, y) \in X \times Y$. Since U has closed range, from the open mapping theorem we have that U is relatively open. Because $T(X) \cap S(X) = \{0\}$, it easily follows that both T and S are open operators. This is equivalent to saying that T and S have closed ranges. ∎

Lemma 2.24. *If $T \in L(X)$ is an operator for which $T(X) \cap \ker T = \{0\}$ and $T(X) + \ker T$ is closed in X, then $T(X)$ is closed. In particular, if $p(T) \leq 1$ and $T(X) + \ker T$ is closed in X, then $T(X)$ is closed.*

Proof. The first assertion immediately follows from Lemma 2.23 applied to the operator T and the natural inclusion mapping from $\ker T$ into X. The second assertion is clear from Theorem 2.7. ∎

Example 2.25. We first show an example for which $T^2(X) = T(X)$ and $T(X)$ is not closed. Let X be a Hilbert space with an orthonormal basis $(e_{i,j})$. Let T be defined by:

$$Te_{i,j} := \begin{cases} 0 & \text{if } j = 1, \\ \frac{1}{i} e_{i,1} & \text{if } j = 2, \\ e_{i,j-1} & \text{otherwise} \end{cases}$$

Let M_1 denote the subspace generated by the set $\{e_{i,j} : j \geq 2, i \geq 1\}$, and M_2 the subspace generated by the set $\{e_{i,2} : i \geq 1\}$. It is easily seen that $T^2(X) = T(X) = M_1 + T(M_2)$. Further, if M_3 denotes the subspace generated by the set $\{e_{i,1} : i \geq 1\}$, the intersection $T(X) \cap M_3$ is not closed, from which we deduce that $T(X)$ is not closed. Therefore, $q(T) \leq 1$ so that T has topological uniform descent. We show that T^* does not have uniform topological descent. In fact, suppose that T^* has topological uniform descent $n \geq d$, then $T^*(X^*) + \ker T^{*d}$ is closed, by part (v) of Theorem 2.16. Since $T^2(X) = T(X)$ we then have $\ker T^* = \ker T^{*2}$, so $p(T^*) \leq 1$. This implies that $T^*(X^*) \cap \ker T^{*d} = \{0\}$ and, by Theorem 2.24, implies that T^* has closed range, or equivalently, $T(X)$ is closed, and this is impossible. Therefore, T^* does not have uniform topological descent.

The class of quasi-Fredholm operator has been first introduced by Labrousse [66], which considered this class in the case of Hilbert spaces operators. Consider the set

$$\Delta(T) := \{n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow T^n(X) \cap \ker\, T \subseteq T^m(X) \cap \ker\, T\}.$$

The *degree of stable iteration* is defined as $\mathrm{dis}(T) := \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\mathrm{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 2.26. *[25, Proposition 2.6] $T \in L(X)$ is said to be* quasi-Fredholm of degree d, *if there exists $d \in \mathbb{N}$ such that:*
(a) *$dis(T) = d$,*
(b) *$T^n(X)$ is a closed subspace of X for each $n \geq d$,*
(c) *$T(X) + \ker\, T^d$ is a closed subspace of X.*

Evidently, the condition (a) entails that $k_n = 0$ for $n \geq d$, so every quasi-Fredholm operator has uniform descent for $n \geq d$.

Theorem 2.27. *If $T \in L(X)$ then the following implications hold:*

$$T \in \Phi_\pm(X) \;\Rightarrow T \text{ quasi-Fredholm } \;\Rightarrow T \text{ has topological uniform descent.}$$

Every essentially semi-regular operators has topological uniform descent.

Proof. Every semi-Fredholm operator has topological uniform descent for $n \geq 0$, so, by Theorem 2.14, $T^n(X) + \ker\, T$ is constant for all $n \geq 1$. Moreover, T^n is semi-Fredholm for all $n \in \mathbb{N}$, hence all $T^n(X)$ are closed. The condition (c) is trivially satisfied, by Theorem 2.16. This shows the first implication. To see the second implication, observe first that, if T is quasi-Fredholm then, always by part (ii) of Theorem 2.14, T has uniform descent. Since the condition (iv) of Theorem 2.16 is satisfied by part (c) of the definition of quasi-Fredholm operators, it then follows that T has topological uniform descent.
■

Let $QF(d)$ denote the class of all quasi-Fredholm of degree d. The following result has a crucial role in the characterizations of quasi-Fredholm operators.

Theorem 2.28. *[74] Let $T \in L(X)$, and suppose that T has finite ascent $p := p(T) < \infty$. Then the following statements are equivalent:*
(i) *there exists $n \geq p + 1$ such that $T^n(X)$ is closed;*
(ii) *$T^n(X)$ is closed for every $n \geq p$;*
(iii) *$T^n(X) + \ker\, T^m$ is closed for all $m, n \in \mathbb{N}$ with $m + n \geq p$.*
Analogous statements hold if T has finite descent $q := q(T) < \infty$.

Dealing with quasi-Fredholm operators, another application of Theorem 2.28 gives a characterization of these operators:

Theorem 2.29. *[25, Corollary 3.3] $T \in L(X)$ is quasi-Fredholm if and only if there exists $p \in \mathbb{N}$ such that $T(X) + \ker\, T^p = T(X) + \mathcal{N}^\infty(T)$ and $T^{p+1}(X)$ is closed.*

Further characterizations of quasi-Fredholm operators are given in the following theorem:

Theorem 2.30. *[25, Proposition 3.2]Suppose that $T \in L(X)$ and $d \in \mathbb{N}$. Then the following statements are equivalent:*

(i) *$T \in QF(d)$;*
(ii) *$dis(T) = d$ and $T^{d+1}(X)$ is closed;*
(iii) *there exists an integer $n \geq 0$ such that $T^n(X)$ is closed and the restriction $T_n := T|T^n(X)$ is semi-regular.*

Evidently, by part (iii) of Theorem 2.30,

$$T \text{ semi-regular} \Rightarrow T \text{ quasi-Fredholm.}$$

We now consider a class of operators, introduced and studied by Berkani *et al.* in a series of papers ([24],[25],[27],[30]) which extends the class of semi-Fredholm operators. For every $T \in L(X)$ and a nonnegative integer n, let us denote by T_n the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set $T_0 = T$).

Definition 2.31. *$T \in L(X)$, X a Banach space, is said to be* B-Fredholm, *(resp.,* semi B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm*), if for some integer $n \geq 0$ the range $T^n(X)$ is closed and T_n is a Fredholm operator (resp., semi-Fredholm, upper semi-Fredholm, lower semi-Fredholm).*

It is easily seen that every nilpotent operator, as well as any idempotent bounded operator, is B-Fredholm. Therefore the class of B-Fredholm operators contains the class of Fredholm operators as a proper subclass.

Theorem 2.32. *[24, Proposition 2.1] Let $T \in L(X)$ and suppose that $T^n(X)$ is closed and T_n is a Fredholm operator (resp., semi-Fredholm, upper semi-Fredholm, lower semi-Fredholm). For every $m \geq n$ then $T^m(X)$ is closed and T_m is a Fredholm operator (resp., semi-Fredholm, upper semi-Fredholm, lower semi-Fredholm), with $ind\, T_{m]} = ind\, T_n$.*

Every semi B-Fredholm operator T has topological uniform descent. Indeed, we show now that every semi B-Fredholm operator is quasi-Fredholm.

Theorem 2.33. *[24, Proposition 2.6] Every semi B-Fredholm operator is quasi Fredholm. More, precisely we have:*

(i) *T is upper semi B-Fredholm if and only if there is an integer d such that $T \in QF(d)$ and $\ker T \cap T^d(X)$ has finite dimension.*

(ii) *T is lower semi B-Fredholm if and only if there is an integer d such that $T \in QF(d)$ and $T(X) + \ker T^d$ has finite codimension.*

(iii) *T is B-Fredholm if and only if there is an integer d such that $T \in QF(d)$, $\ker T \cap T^d(X)$ has finite dimension and $T(X) + \ker T^d$ has finite codimension.*

The following punctured disc theorem is obtained by combining Corollary 2.20 and Theorem 2.33, in the special case that $S = \lambda I - T$, with λ sufficiently small.

Theorem 2.34. *Suppose that $T \in L(X)$ is upper semi B-Fredholm. Then there exists an open disc $\mathbb{D}(0, \varepsilon)$ centered at 0 such that $\lambda I - T$ is upper semi-Fredholm for all $\lambda \in \mathbb{D}(0, \varepsilon) \setminus \{0\}$ and*

$$ind\,(\lambda I - T) = ind\,(T) \quad \textit{for all } \lambda \in \mathbb{D}(0, \varepsilon).$$

Moreover, if $\lambda \in \mathbb{D}(0, \varepsilon) \setminus \{0\}$ then

$$\alpha(\lambda I - T) = dim\,(\ker T \cap T^d(X)) \quad \textit{for some } d \in \mathbb{N},$$

so that $\alpha(\lambda I - T)$ is constant as λ ranges in $\mathbb{D}(0, \varepsilon) \setminus \{0\}$ and

$$\alpha(\lambda I - T) \leq \alpha(T) \quad \textit{for all } \lambda \in \mathbb{D}(0, \varepsilon).$$

Analogously, if $T \in L(X)$ is lower semi B-Fredholm then there exists an open disc $\mathbb{D}(0, \varepsilon)$ centered at 0 such that $\lambda I - T$ is lower semi-Fredholm for all $\lambda \in \mathbb{D}(0, \varepsilon) \setminus \{0\}$ and

$$ind\,(\lambda I - T) = ind\,(T) \quad \textit{for all } \lambda \in \mathbb{D}(0, \varepsilon).$$

Moreover, if $\lambda \in \mathbb{D}(0, \varepsilon) \setminus \{0\}$ then

$$\beta(\lambda I - T) = codim\,(\ker T^d + T(X)) \quad \textit{for some } d \in \mathbb{N},$$

so that $\beta(\lambda I - T)$ is constant as λ ranges in $\mathbb{D}(0, \varepsilon) \setminus \{0\}$ and

$$\beta(\lambda I - T) \leq \beta(T) \quad \textit{for all } \lambda \in \mathbb{D}(0, \varepsilon).$$

Evidently, Theorem 2.34 is an extension of the classical punctured neighborhood theorem for semi-Fredholm operators, to semi B-Fredholm operators.

B-Fredholm operators on Banach spaces may be characterized through a decomposition which similar to the Kato decomposition (but recall that a Fredholm operator may be not semi-regular):

Theorem 2.35. *[27, Lemma 4.1] Let $T \in L(X)$, X a Banach space. Then*

(i) *T is B-Fredholm if and only if there exist two closed invariant subspaces M and N such that $X = M \oplus N$, $T|M$ is Fredholm and $T|N$ is nilpotent.*

(ii) *T is B-Fredholm of index 0 if and only if there exist two closed invariant subspaces M and N such that $X = M \oplus N$, $T|M$ is Fredholm having index 0 and $T|N$ is nilpotent.*

According Caradus [35] an operator $T \in L(X)$ is said to be *generalized Fredholm* if there exists an operator $S \in L(X)$ such that $TST = T$ and $I - ST - TS \in \Phi(X)$. Examples of generalized Fredholm operators are projections, finite-dimensional and Fredholm operators. This class of operators has been studied in several papers by Schmoeger ([88], [89], [90], [91], [92], [93]) which shows that T is generalized Fredholm if and only if there exist two closed invariant subspaces M and N such that $X = M \oplus N$, $T|M$ is Fredholm and $T|N$ is a finite rank nilpotent operator, see [92, Theorem 1.1]. Therefore, by Theorem 2.35 every generalized Fredholm operator is B-Fredholm, but the converse is not true. For instance, a nilpotent operator with non-closed range is a B-Fredholm operator but not a generalized Fredholm operator. The relationship between B-Fredholm operators and generalized Fredholm operators is fixed by the following theorem.

Theorem 2.36. *$T \in L(X)$ is B-Fredholm if and only if there exists $n \in \mathbb{N}$ such that T^n is a generalized Fredholm operator.*

The following concept of invertibility has been introduced in [84] and was inspired by the work of Drazin [42].

Definition 2.37. *Let $\mathcal{A}$ be an algebra with unit e. An element $a \in \mathcal{A}$ is said to be a* Drazin invertible element of degree *n if there is an element $b \in \mathcal{A}$ such that*

$$a^n ba = a^n, \quad bab = b, \quad ab = ba. \tag{11}$$

The element b is called the Drazin inverse *(of degree n). If $a \in \mathcal{A}$ satisfies the equalities (11) for $k = 1$ then a is called* group invertible.

In the following remark we shall reassume some basic properties of group invertible elements.

Remark 2.38. (a) An element a can have only one Drazin inverse. of the same degree. Every Drazin inverse of degree n is also a Drazin inverse of degree $n + 1$, so if a has Drazin inverse b of degree n and a Drazin inverse c of degree k, the $b = c$. Moreover, $a \in \mathcal{A}$ is Drazin invertible of degree n if and only if a^n is group invertible in $\mathcal{A}$, see [84, Lemma 1 and Corollary 5].

(b) An element $a \in \mathcal{A}$ is group invertible if and only if a admits a commuting *generalized inverse*, i.e. there is $b \in \mathcal{A}$ such that $ab = ba$ and $aba = a$, or equivalently there exists $b \in \mathcal{A}$ such that $aba = a$ and $e - ab - ba$ is invertible in $\mathcal{A}$, see [88, Theorem 3.3 and Proposition 3.9].

(c) If $a, b \in \mathcal{A}$ are two commuting Drazin invertible elements then ab is Drazin invertible, see [31, Proposition 2.6].

Let $\mathcal{F}(X)$ denote the two-sided ideal of all finite dimensional operators in $L(X)$, and denote by $\mathcal{L}$ the normed algebra $L(X)/\mathcal{F}(X)$ provided with the canonical quotient norm. Let $\pi : L(X) \to \mathcal{L}$ the canonical quotient mapping. The well known Atkinson's theorem say that $T \in \Phi(X)$ if and only if $\pi(T)$ is invertible in $\mathcal{L}$. A version of Atkinson' s theorem may be stated for B-Fredholm theory as follows:

Theorem 2.39. *[31] $T \in L(X)$ is B-Fredholm if and only if $\pi(T)$ is a Drazin invertible element of the algebra $\mathcal{L}$.*

The next result shows that a characterization, similar to that of Theorem 2.35 established for Fredholm operators on Banach spaces, also holds for semi B-Fredholm operators in the case of *Hilbert space operators*.

Theorem 2.40. *If $T \in L(H)$, H a Hilbert space, then T is upper semi B-Fredholm (respectively, lower semi B-Fredholm) if and only if there exist two closed T-invariant subspaces M and N such that $X = M \oplus N$, $T|M$ is upper semi-Fredholm (respectively, lower semi Fredholm) and $T|N$ is nilpotent.*

A bounded operator T which is Drazin invertible in $\mathcal{A} := L(X)$ is simply said to be Drazin invertible. Drazin invertibility for operators may be characterized in several way:

Theorem 2.41. *[78, Theorem 10, Chapter 3] If $T \in L(X)$ then the following statements are equivalent:*

(i) *T is Drazin invertible, i.e. there is $S \in L(X)$ such that $TS = ST$, $STS = S$ and $T^nST = T^n$;*

(ii) *there is $S \in L(X)$ which commutes with T and $n \in \mathbb{N}$ such that $T^{n+1}S = T^n$.*

(iii) *$p(T) = q(T) \leq n$;*

(iv) *$T = T_1 \oplus T_2$, where T_1 is nilpotent and T_2 is invertible;*

(v) *Either $0 \notin \sigma(T)$ or 0 is an isolated point of $\sigma(T)$ and the restriction of T onto the range $P(X)$ of the spectral projection associated at $\{0\}$ is nilpotent.*

The concept of Drazin invertibility suggests the following definition:

Definition 2.42. *$T \in L(X)$, X a Banach space, is said to be* left Drazin invertible *if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed, while $T \in L(X)$ is said to be* right Drazin invertible *if $q := q(T) < \infty$ and $T^q(X)$ is closed.*

The operator considered in Example 2.25 shows that I the condition $q = q(T) < \infty$ does not entails that $T^q(X)$ is closed. Clearly, $T \in L(X)$ is both right and left Drazin invertible if and only if T is Drazin invertible.

The next result characterizes the left Drazin invertible and the right Drazin invertible operators among the operators which have topological uniform descent.

Theorem 2.43. *[19] Suppose that $T \in L(X)$. Then the following statements are equivalent:*

(i) *T is left Drazin invertible;*

(ii) *T is quasi-Fredholm and has finite ascent;*

(iii) *T has topological uniform descent and has finite ascent.*

Dually, following statements are equivalent:

(iv) *T is right Drazin invertible;*

(v) *T is quasi-Fredholm and has finite descent:*

(vi) *T has topological uniform descent and has finite descent.*

By Theorem 2.7, if T is Fredholm operator with index 0 and either $p(T)$ or $q(T)$ is finite then T is Browder. This result, in the framework of B-Fredholm theory, may be generalized as follows:

Theorem 2.44. *[12, Theorem 2.2] For an operator $T \in L(X)$ the following statements hold:*

(i) *If T is upper semi B-Fredholm with index $\text{ind}\, T \leq 0$ and $q(T) < \infty$, then T is Drazin invertible.*

(ii) *If T is lower semi B-Fredholm with index $\text{ind}\, T \geq 0$ and $p(T) < \infty$, then T is Drazin invertible.*

(iii) *If T is B-Fredholm with index $\text{ind}\, T = 0$ and has either ascent or descent finite, then T is Drazin invertible.*

We now show that the concept of left and right Drazin invertibility are dual each other:

Theorem 2.45. *[18] For every $T \in L(X)$ the following equivalences hold:*

(i) *T is left Drazin invertible $\Leftrightarrow$ T^* is right Drazin invertible.*

(ii) *T is right Drazin invertible $\Leftrightarrow$ T^* is left Drazin invertible.*

(iii) *T is Drazin invertible if and only if T^* is Drazin invertible.*

3. The localized SVEP and Fredholm theory

We have seen in Theorem 1.9 that the SVEP for T holds precisely when for every element $0 \neq x \in X$ we have $\sigma_T(x) = \emptyset$. The next fundamental theorem, which establishes a localized version of this result,will be useful in the sequel .

Theorem 3.1. *[1, Theorem 2.22] If $T \in L(X)$ the following statements are equivalent:*

(i) *T has SVEP at λ_0;*

iii) $\ker(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$;

(iii) *For each $0 \neq x \in \ker(\lambda_0 I - T)$ we have $\sigma_T(x) = \{\lambda_0\}$.*

It should be noted that

$$\ker(\lambda_0 I - T) \cap X_T(\emptyset) = \{0\} = \ker(\lambda_0 I - T) \cap K(\lambda_0 I - T)$$

holds for every $T \in L(X)$. To see this observe first that by Theorem 1.14 we have $\ker T \subseteq H_0(T) \subseteq X_T(\{0\})$. From Theorem 1.21 it follows that

$$\ker T \cap K(T) = \ker T \cap X_T(\mathbb{C} \setminus \{0\}) \subseteq X_T(\{0\}) \cap X_T(\mathbb{C} \setminus \{0\}) = X_T(\emptyset).$$

Since $X_T(\emptyset) \subseteq X_T(\mathbb{C} \setminus \{0\}) = K(T)$ we then conclude that

$$\ker T \cap K(T) = \ker T \cap K(T) \cap X_T(\emptyset) = \ker T \cap X_T(\emptyset).$$

For an arbitrary operator $T \in L(X)$ on a Banach space X let

$$\Xi(T) := \{\lambda \in \mathbb{C} : T \text{ fails to have SVEP at } \lambda\} .$$

Clearly, $\Xi(T)$ is contained in the interior of the spectrum $\sigma(T)$, and, from the identity theorem for analytic functions it readily follows that $\Xi(T)$ is open. Clearly $\Xi(T)$ is empty precisely when T has the SVEP.

Corollary 3.2. *If $T \in L(X)$ is surjective, then T has SVEP at 0 if and only if T is injective. Consequently, the equality $\sigma(T) = \sigma_{\rm s}(T) \cup \Xi(T)$ holds for every $T \in L(X)$. Furthermore, $\sigma_{\rm su}(T)$ contains $\partial\Xi(T)$, the topological boundary of $\Xi(T)$.*

Proof. If T is onto and has SVEP at 0 then $K(T) = X$. By Theorem 3.1 we have $\ker T \cap X = \ker T = \{0\}$, hence T is injective. The converse is clear. To show the equality $\sigma(T) = \sigma_{\rm s}(T) \cap \Xi(T)$ we have only to show the inclusion ($\subseteq$). Suppose that $\lambda \notin \sigma_{\rm s}(T) \cap \Xi(T)$. From the first part we obviously have $\lambda \notin \sigma(T)$ from which we obtain $\sigma(T) = \sigma_{\rm s}(T) \cup \Xi(T)$. The last claim is immediate: since $\partial\Xi(T) \subseteq \sigma(T)$ and $\Xi(T)$ is open it then follows that $\partial\Xi(T) \cap \Xi(T) = \emptyset$. This obviously implies that $\partial\Xi(T) \subseteq \sigma_{\rm su}(T)$. ∎

Let L denote the unilateral left shift on the Hilbert space $\ell_2(\mathbb{N})$, defined as

$$L(x_1, x_2, x_3, \cdots) := (x_2, x_3, \cdots) \quad \text{for all } x = (x_n) \in \ell_2(\mathbb{N}).$$

Evidently, L is onto but not injective, since every vector $(x_1, 0, 0, \cdots)$, with $x_1 \neq 0$, belongs to $\ker L$. Corollary 3.2 then shows that L fails to have SVEP at 0. Later, we shall see that other examples of operators which do not have SVEP at 0 are semi-Fredholm operators on a Banach space having index strictly greater than 0.

Theorem 3.3. *[1, Theorem 3.8] For a bounded operator T on a Banach space X and $\lambda_0 \in \mathbb{C}$, the following implications hold:*

$$\begin{aligned} p(\lambda_0 I - T) < \infty \quad &\Rightarrow \quad \mathcal{N}^\infty(\lambda_0 I - T) \cap (\lambda_0 I - T)^\infty(X) = \{0\} \\ &\Rightarrow \quad T \text{ has the SVEP at } \lambda_0, \end{aligned}$$

and

$$\begin{aligned} q(\lambda_0 I - T) < \infty \quad &\Rightarrow \quad X = \mathcal{N}^\infty(\lambda_0 I - T) + (\lambda_0 I - T)^\infty(X) \\ &\Rightarrow \quad T^\star \text{ has the SVEP at } \lambda_0. \end{aligned}$$

In the remaining part of this section we want show that the relative positions of all the subspaces introduced in the previous chapter are intimately related to the SVEP at a point.

To see that let us consider, for an arbitrary $\lambda_0 \in \mathbb{C}$ and an operator $T \in L(X)$ the following increasing chain of kernel type of spaces:

$$\ker(\lambda_0 I - T) \subseteq \mathcal{N}^\infty(\lambda_0 I - T) \subseteq H_0(\lambda_0 I - T) \subseteq X_T(\{\lambda_0\}),$$

and the decreasing chain of the range type of spaces:

$$X_T(\emptyset) \subseteq X_T(\mathbb{C} \setminus \{\lambda_0\}) = K(\lambda_0 I - T) \subseteq (\lambda_0 I - T)^\infty(X) \subseteq (\lambda_0 I - T)(X).$$

The next corollary is an immediate consequence of Theorem 3.1 and the inclusions considered above.

Corollary 3.4. *Suppose that $T \in L(X)$ verifies one of the following conditions:*
(i) $\mathcal{N}^\infty(\lambda_0 I - T) \cap (\lambda_0 I - T)^\infty(X) = \{0\}$;
(ii) $\mathcal{N}^\infty(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$;
(iii) $\mathcal{N}^\infty(\lambda_0 I - T) \cap X_T(\emptyset) = \{0\}$;
(iv) $H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$;
(v) $\ker(\lambda_0 I - T) \cap (\lambda_0 I - T)(X) = \{0\}$.
Then T has the SVEP at λ_0. ∎

Theorem 3.5. *[1, Cor 2.45] For $T \in L(X)$, the following statements hold:*
(i) *If T has the SVEP then $\sigma_{\mathrm{su}}(T) = \sigma(T)$ and $\sigma_{\mathrm{se}}(T) = \sigma_{\mathrm{ap}}(T)$.*
(ii) *If $T^\star$ has the SVEP then $\sigma_{\mathrm{ap}}(T) = \sigma(T)$ and $\sigma_{\mathrm{se}}(T) = \sigma_{\mathrm{su}}(T)$.*
(iii) *If both T and $T^\star$ have the SVEP then*

$$\sigma(T) = \sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{se}}(T).$$

Then converse of Corollary 3.4 need not be true. The next bilateral weighted shift, provides an example of operator T which has SVEP at 0 while $H_0(T) \cap K(T) \neq \{0\}$.

Example 3.6. *[8]* Let $\beta := (\beta_n)_{n\in\mathbb{Z}}$ be the sequence of real numbers defined as follows:

$$\beta_n := \begin{cases} 1 + |n| & \text{if } n < 0, \\ e^{-n^2} & \text{if } n \geq 0. \end{cases}$$

Let $X := L_2(\beta)$ denote the Hilbert space of all formal Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n \quad \text{for which} \quad \sum_{n=-\infty}^{\infty} |\alpha_n|^2 {\beta_n}^2 < \infty\,,$$

Let us consider the *bilateral weighted right shift* defined by

$$T(\sum_{n=-\infty}^{\infty} a_n z^n) := \sum_{n=-\infty}^{\infty} a_n z^{n+1} ,$$

or equivalently, $Tz^n := z^{n+1}$ for every $n \in \mathbb{Z}$. The operator T is bounded on $L_2(\beta)$ and

$$\|T\| = \sup \left\{ \frac{\beta_{n+1}}{\beta_n} : n \in \mathbb{Z} \right\} = 1 .$$

Clearly T is injective, so it has the SVEP at 0. We show now that $H_0(T) \cap K(T) \neq \{0\}$. From $\|z^n\|_\beta = \beta_n$ for all $n \in \mathbb{Z}$ we obtain that

$$\lim_{n\to\infty} \|z^{n-1}\|_\beta^{1/n} = 0$$

and

$$\lim_{n\to\infty} \|z^{-n-1}\|_\beta^{1/n} = 1 .$$

By the formula for the radius of convergence of a power series we then conclude that the two series

$$f(\lambda) := \sum_{n=1}^{\infty} \lambda^{-n} z^{n-1} \quad \text{and} \quad g(\lambda) := -\sum_{n=1}^{\infty} \lambda^{n} z^{-n-1}$$

converge in $L_2(\beta)$ for all $|\lambda| > 0$ and $|\lambda| < 1$, respectively. Evidently, the function f is analytic on $\mathbb{C} \setminus \{0\}$, and

$$(\lambda I - T) f(\lambda) = -\sum_{n=1}^{\infty} \lambda^{-n} z^{n} - \sum_{n=1}^{\infty} \lambda^{1-n} z^{n-1} = 1 \quad \text{for all } \lambda \neq 0 ,$$

while the function g, which is analytic on the open unit disc $\mathbb{D}(0, 1)$, verifies

$$(\lambda I - T) g(\lambda) = \sum_{n=0}^{\infty} \lambda^{n} z^{-n} - \sum_{n=0}^{\infty} \lambda^{1+n} z^{-n-1} = 1 \quad \text{for all } \lambda \in \mathbb{D}(0, 1) .$$

This implies that $1 \in \mathcal{X}_T(\{0\}) \cap \mathcal{X}_T(\mathbb{C} \setminus \mathbb{D}(0, 1)) = H_0(T) \cap K(T)$, where the last equality follows from Theorem 1.21 and Theorem 1.14.

For a quasi-nilpotent operator we have $H_0(T) = X$. The analytic core $K(T)$ is "near" to be the complement of $H_0(T)$. Indeed we have:

Corollary 3.7. *If $T \in L(X)$ is quasi-nilpotent then $K(T) = \{0\}$.*

We now establish some other conditions that entail SVEP:

Theorem 3.8. *[1, Theorem 2.31] For a bounded operator $T \in L(X)$, X a Banach space, the following implications hold :*

(i) *$H_o(\lambda_0 I - T)$ closed $\Rightarrow H_o(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\} \Rightarrow T$ has the SVEP at λ_0.*

(ii) *$X = H(\lambda_0 I - T) + K(\lambda_0 I - T) \Rightarrow T^\star$ has the SVEP at λ_0.*

The operator in the Example 1.17 shows that, in general, the converse of Theorem 3.8 does not hold. Indeed, T has SVEP, since the point spectrum $\sigma_{\mathrm{p}}(T)$ is empty, while $H(T)$ is not closed.

Theorem 3.9. *Suppose that for a bounded operator $T \in L(X)$, the sum $H_0(\lambda_0 I - T) + (\lambda_0 I - T)(X)$ is norm dense in X. Then $T^\star$ has the SVEP at λ_0.*

Corollary 3.10. *Suppose either that $H_0(\lambda_0 I - T) + K(\lambda_0 I - T)$ or $\mathcal{N}^\infty(\lambda_0 I - T) + (\lambda_0 I - T)^\infty(X)$ is norm dense in X. Then $T^\star$ has the SVEP at λ_0.* ∎

Also the result of Corollary 3.10 cannot be reversed, as the following example shows:

Example 3.11. Let V denote the *Volterra operator* on the Banach space $X := C[0, 1]$, defined by

$$(Vf)(t) := \int_0^t f(s)\mathrm{d}s \quad \text{for all } f \in C[0,1] \quad \text{and } t \in [0,1].$$

V is injective and quasi-nilpotent. Consequently $\mathcal{N}^\infty(V) = \{0\}$ and $K(V) = \{0\}$ by Corollary 3.7. It is easy to check that

$$V^\infty(X) = \{f \in C^\infty[0,1] : f^{(n)}(0) = 0,\ n \in \mathbb{Z}_+\},$$

thus $V^\infty(X)$ is not closed and hence is strictly larger than $V(T) = \{0\}$. Clearly the sum $\mathcal{N}^\infty(V) + V^\infty(X)$ is not norm dense in X, while $V^\star$ has the SVEP, because it is quasi-nilpotent.

Given an operator $T \in L(X)$, X a Banach space, and an analytic function f defined on an open neighborhood $\mathcal{U}$ of $\sigma(T)$, and let $f(T)$ denote the corresponding operator defined by the functional calculus. One may be tempted to conjecture that the spectral theorem for the local spectrum, i.e. $f(\sigma_T(x)) = \sigma_{f(T)}(x)$ for all $x \in X$. It can be easily seen that in general that this equality is not true. Indeed, if we consider the constant function $f \equiv c$ on the neighborhood $\mathcal{U}$ and an operator T without the SVEP, then there exists, by Theorem 1.21, a vector $0 \neq x \in X$ such that $\sigma_T(x) = \emptyset$. Clearly $f(\sigma_T(x) = \emptyset$, while

$$\sigma_{f(T)}(x) = \sigma(f(T)) = \{c\} \neq \emptyset.$$

Denote by $\mathcal{H}(\sigma(T))$ the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$. In order to show that the spectral theorem for the local spectrum holds if T has SVEP, we need to prove before that the SVEP is preserved under the functional calculus.

Theorem 3.12. *[68, Theorem 3.3.8] If $T \in L(X)$ has SVEP then $f(T)$ has SVEP for every $f \in \mathcal{H}(\sigma(T))$.*

Denote by $\mathcal{H}_{\rm nc}(\sigma(T))$ the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is nonconstant on each of the components of its domain.

Theorem 3.13. *[68, Theorem 3.3.8] If $T \in L(X)$ the following statements hold:*
(i) *$f(\sigma_T(x)) \subseteq \sigma_{f(T)}(x)$ for all $x \in X$ and $f \in \mathcal{H}(\sigma(T))$.*
(ii) *If T has SVEP, or if the function $f \in \mathcal{H}_{\rm nc}(\sigma(T))$, then*

$$f(\sigma_T(x)) = \sigma_{f(T)}(x) \quad \textit{for all } x \in X,$$

The next result shows that the localized SVEP is preserved under the functional calculus under appropriate condition on the analytic function.

Theorem 3.14. *[8] Let $T \in L(X)$ and $f \in \mathcal{H}_{\rm nc}(\sigma(T))$. Then $f(T)$ has the SVEP at $\lambda \in \mathbb{C}$ if and only if T has the SVEP at every point $\mu \in \sigma(T)$ for which $f(\mu) = \lambda$.*

Combining Theorem 3.14 and Theorem 3.12 we then have:

Corollary 3.15. *Let $T \in L(X)$ and $f \in \mathcal{H}_{\rm nc}(\sigma(T))$. Then T has the SVEP if and only if $f(T)$ has the SVEP.*

An immediate consequence of Theorem 3.14 is that, in the characterization of the SVEP at a point $\lambda_0 \in \mathbb{C}$ given in Theorem 3.1, the kernel $\ker(\lambda_0 I - T)$ may be replaced by the hyper-kernel $\mathcal{N}^\infty(\lambda_0 I - T)$.

Corollary 3.16. *For every bounded operator on a Banach space X the following properties are equivalent:*

(i) *T has the SVEP at λ_0;*

(ii) *$(\lambda_0 I - T)^n$ has the SVEP at 0 for every $n \in \mathbb{N}$.*

(iii) *$\mathcal{N}^\infty(\lambda_0 I - T) \cap X_T(\emptyset) = \{0\}$;*

(iv) *$\mathcal{N}^\infty(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$.*

Note that in the condition (ii) of Corollary 3.16 the power $(\lambda_0 I - T)^n$ may be replaced by $f(T)$, where f is any analytic function on some neighborhood $\mathcal{U}$ of $\sigma(T)$ such that f is non-constant on each of the connected components of $\mathcal{U}$ and such that 0 is the only zero of f in $\sigma(T)$.

The following result generalizes Corollary 3.2 to semi-regular operators, since every surjective operator is semi-regular.

Theorem 3.17. *Suppose that $\lambda_0 I - T$ is a semi-regular operator on the Banach space X. Then the following equivalences hold:*

(i) *T has the SVEP at λ_0 precisely when $\lambda_0 I - T$ is injective or, equivalently, when $\lambda_0 I - T$ is bounded below;*

(ii) *$T^\star$ has the SVEP at λ_0 precisely when $\lambda_0 I - T$ is surjective.*

Proof. (i) We can assume that $\lambda_0 = 0$. We have only to prove that if T has the SVEP at 0 then T is injective. Suppose that T is not injective. Then, by Theorem 1.23, the semi-regularity of T entails $T^\infty(X) = K(T)$ and $\{0\} \neq \ker T \subseteq T^\infty(X) = K(T)$. Thus T does not have the SVEP at 0 by Theorem 3.1.

(ii) If $\lambda_0 I - T$ is semi-regular then also $\lambda_0 I^\star - T^\star$ is semi-regular and $\lambda_0 I - T$ is surjective if and only if $\lambda_0 I^\star - T^\star$. is bounded below . ∎

Now we show that the SVEP at a point for operators having topological uniform descent may be characterized in several ways. These characterizations extend previous results obtained in [2] for quasi-Fredholm operators.

Theorem 3.18. *[61] Suppose that $\lambda_0 I - T$ has topological uniform descent for $n \geq d$. Then the following conditions are equivalent:*

(i) *T has SVEP at λ_0;*

(ii) *the restriction $T|(\lambda_0 I - T)^d(X)$ has SVEP at λ_0, where the subspace $(\lambda_0 I - T)^d(X)$ is equipped with the operator range topology;*

(iii) *the restriction $(\lambda_0 I - T)|(\lambda_0 I - T)^d(X)$ is bounded below, where $(\lambda_0 I - T)^d(X)$ is equipped with the operator range topology;*

(iv) *$\lambda_0 I - T$ has finite ascent, or equivalently the operator $\lambda_0 I - T$ is left Drazin invertible;*

(v) *$\sigma_{\rm ap}(T)$ does not cluster at λ_0;*

(vi) *$\lambda_0 \notin int\, \sigma_{\rm ap}(T)$, where $int\, \sigma(T)$ is the interior of $\sigma_{\rm ap}(T)$;*

(vii) *there exists $p \in \mathbb{N}$ such that $H_0(\lambda_0 I - T) = \ker (\lambda_0 I - T)^p$;*

(viii) *$H_0(\lambda_0 I - T)$ is closed;*

(ix) *$H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$;*

(x) *$\mathcal{N}^\infty(\lambda_0 I - T) \cap (\lambda_0 I - T)^\infty(X) = 0\}$.*

In particular, these equivalences hold for semi B-Fredholm operators.

Next, we will consider some characterizations of SVEP for T^* at λ_0 in the case that $\lambda_0 I - T$ has topological uniform descent. Recall that the property of having topological uniform descent is not transmitted by duality, so we cannot use the results of Theorem 3.18.

Theorem 3.19. *Suppose that $\lambda_0 I - T$ has topological uniform descent for $n \geq d$. Then the following conditions are equivalent:*

(i) *T^* has SVEP at λ_0;*

(ii) *there exists $n \in \mathbb{N}$ such that restriction $(\lambda_0 I - T)|(\lambda_0 I - T)^n(X)$ is onto, where $(\lambda_0 I - T)^n(X)$ is equipped with the operator range topology;*

(iii) *$\lambda_0 \notin \mathrm{int}\, \sigma_s(T)$;*

(iv) *$\sigma_s(T)$ does not cluster at λ_0;*

(v) *$\lambda_0 I - T$ has finite descent, or equivalently the operator $\lambda_0 I - T$ is right Drazin invertible;*

(vi) *there exists $q \in \mathbb{N}$ such that $K(\lambda_0 I - T) = (\lambda_0 I - T)^q(X)$;*

(vii) *$X = H_0(\lambda I - T) + K(\lambda_0 I - T)$;*

(viii) *$H_0(\lambda I - T) + K(\lambda_0 I - T)$ is norm dense in X;*

(ix) *$X = \mathcal{N}^\infty(\lambda_0 I - T) + (\lambda_0 I - T)^\infty(X)$;*

(x) *$\mathcal{N}^\infty(\lambda_0 I - T) + (\lambda_0 I - T)^\infty(X)$ is norm dense in X.*

In particular, the equivalences hold for semi B-Fredholm operators.

The next corollary is an obvious consequence of Theorem 3.18 and Theorem 3.19.

Corollary 3.20. *[61] If $\lambda_0 I - T$ has uniform topological descent then the following statements are equivalent:*

(i) *Both T and T^* have SVEP at λ_0;*

(ii) *λ_0 is a pole of the resolvent;*

(iii) *$X = H_0(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$;*

(iv) *$X = \mathcal{N}^\infty(\lambda_0 I - T) \oplus (\lambda_0 I - T)^\infty(X)$.*

Since every essentially semi-regular operator has topological uniform descent, so the results established in Theorem 3.18 and Theorem 3.19 are valid for this class of operators.

Theorem 3.21. *[1, Theorem 3.16 and 3.17] Suppose that $\lambda_0 I - T$ is essentially semi-regular. Then we have*

(i) *T has SVEP at λ_0 if and only if $H_0(\lambda_0 I - T)$ is finite dimensional. In this case $\lambda_0 I - T \in \Phi_+(X)$.*

(ii) *T^* has SVEP at λ_0 if and only if $K(\lambda_0 I - T)$ is finite codimensional. In this case $\lambda_0 I - T \in \Phi_-(X)$.*

For semi-Fredholm operator we have the following important result.

Corollary 3.22. *Suppose that $\lambda_0 I - T$ is semi-Fredholm. Then the following statements hold:*

(i) *If T has SVEP at λ_0 then $\mathrm{ind}\,(\lambda_0 I - T) \leq 0$.*

(ii) *If T^* has SVEP at λ_0 then $\mathrm{ind}\,(\lambda_0 I - T) \geq 0$.*

Consequently, if both T and T^* have SVEP at λ_0 then $\lambda_0 I - T \in \Phi(X)$ and $\mathrm{ind}\,(\lambda_0 I - T) = 0$.

Proof. (i) By Theorem 3.18 we know that if T has SVEP at λ_0 then $p(\lambda_0 I - T) < \infty$, and this implies, by part (i) of Theorem 2.7, that $\alpha(\lambda_0 I - T) \leq \beta(\lambda_0 I - T)$, so $\mathrm{ind}\,(\lambda_0 I - T) \leq 0$.

(ii) By Theorem 3.19 we know that if T^* has SVEP at λ_0 then $q(\lambda_0 I - T) < \infty$, and this implies, by part (i) of Theorem 2.7, that $\beta(\lambda_0 I - T) \leq \alpha(\lambda_0 I - T)$, so $\mathrm{ind}\,(\lambda_0 I - T) \geq 0$.

The last assertion is clear. ∎

The converse of the results of Corollary 3.22 do not hold, i.e. a semi-Fredholm operator with index less or equal to 0 may fails SVEP at 0. For instance, if R and L denote the right shift and the left shift on the Hilbert space $\ell_2(\mathbb{N})$, then is is easy to see that R is injective, and $\alpha(R \oplus L) = \alpha(L) = 1$ while $\beta(R \oplus L) = \beta(R) = 1$, so $R \oplus L$ is a Fredholm operator having index 0. Since L fails SVEP at 0, then $R \oplus L$ does not have SVEP at 0.

Corollary 3.23. *[2] If $\lambda I - T \in L(X)$ is B-Fredholm with ind $(\lambda I - T) = 0$ the the following statements are equivalent:*

(i) *T has SVEP at λ;*

(ii) *$\lambda I - T$ is Drazin invertible;*

(iii) *T^* has SVEP at λ.*

Proof. (i) ⇔ (ii) The SVEP of T at λ is equivalent to $p(\lambda I - T) < \infty$, by Theorem 3.18, and this is equivalent to saying that $\lambda I - T$ is Drazin invertible, by Theorem 2.44.

(iii) ⇔ (ii) The SVEP of T^* at λ is equivalent to $q(\lambda I - T) < \infty$, by Theorem 3.19, and this is equivalent to saying that $\lambda I - T$ is Drazin invertible, always by Theorem 2.44. ∎

4. Localized SVEP under commuting perturbations

We first observe that SVEP is not preserved under non-commuting perturbations. In fact, by [96, Example 5.6.29], the sum of a decomposable operator and a rank-one operator may fail to have SVEP, although decomposable operators and finite rank operators have SVEP.

In general the SVEP is also not stable under arbitrary sums and products of commuting operators. A specific example based on the theory of weighted shifts may be found in [33], but here we present a general principle that shows that such examples exist in abundance.

Theorem 4.1. *[16] Let $S \in L(X)$ and suppose that there exist $\alpha \neq \beta \in \mathbb{C}$ such that*

$$K(\alpha I - S) = K(\beta I - S) = \{0\} \tag{12}$$

If $T \in L(X)$ commutes with S, then T is the sum of two commuting operators with SVEP, while $\exp(T)$ *is the product of two commuting operators with SVEP.*

Proof. Since all quasi-nilpotent operators share SVEP, we may assume that the spectral radius $r(T) > 0$. To verify that $T(S - \alpha I)$ has SVEP, we consider an arbitrary open set $U \subseteq \mathbb{C}$ and an analytic function $f : U \to X$ for which

$$(\mu I - T(S - \alpha I))f(\mu) = 0 \quad \text{for all } \mu \in U.$$

For a fixed non-zero $\mu \in U$ and arbitrary $\lambda \in \mathbb{C}$ with $\lambda < |\mu|/r(T)$, the operator $\lambda T - \mu I$ is invertible and its inverse commutes with both S and T. Moreover,

$$\begin{aligned} & (\lambda I - (S - \alpha I))T(\lambda T - \mu I)^{-1} f(\mu) \\ = \quad & (\lambda T - \mu I)^{-1}[(\mu I - T(S - \alpha I)) + (\lambda T - \mu I)]f(\mu) = f(\mu). \end{aligned}$$

This shows that $0 \in \rho_{S-\alpha I}(f(\mu))$ and therefore, by Theorem 1.21,

$$f(\mu) \in K(S - \alpha I) = \{0\} \quad \text{for all } \mu \in U.$$

Thus $f \equiv 0$ on U, which establishes SVEP for $T(S - \alpha I)$ and, of course, similarly also for $T(S - \beta I)$. Because

$$(\beta - \alpha)T = T(S - \alpha I) + T(\beta I - S),$$

the first assertion is now immediate, and the last clam, concerning the product, follows from the fact that SVEP is preserved under the analytical calculus. ∎

Note that in Theorem 4.1 the operators T and $\exp(T)$ may fail to have SVEP, while the condition on S entails that $X_T(\emptyset) = \{0\}$ and hence, by Theorem 1.11, the SVEP for S. To provide concrete examples of operators that satisfy the condition (12) on $K(\lambda I - S)$ of the preceding result, we now introduce the concept of semi-shift.

A bounded operator $S \in L(X)$ is said to a *semi-shift*, if S is an isometry for which $\mathcal{S}^\infty(X) = \{0\}$. Examples of semi-shifts T are the unilateral right shift operators of arbitrary multiplicity on the sequence spaces $\ell^p(\mathbb{N})$, with $1 \leq p < \infty$, defined as

$$Tx := (0, x_1, x_2, \dots) \quad \text{for all } x = (x_n) \in \ell_p(\mathbb{N}).$$

Other important examples of semi-shifts are the *right translation* operators on the Lebesgue spaces $L^p([0, +\infty])$, $1 \leq p < \infty$. Note that if T is a semi-shift then $\sigma_T(x) = \sigma(T)$ coincides with the closed unit disc $\mathbf{D}(0, 1)$ of $\mathbb{C}$ for all non-zero $x \in X$, see [68, Proposition 1.6.8]. Now, if $x \neq 0$ and $\alpha \in \mathbf{D}(0, 1)$ then $\alpha \in \sigma_T(x)$ and hence $0 \in \sigma_{\alpha I - T}(x)$, so that $x \notin K(\alpha I - T)$, by Theorem 1.21. Therefore, $K(\alpha I - T) = \{0\}$ for all $\alpha \in \mathbf{D}(0, 1)$.

To find an operator without SVEP that commutes with a semi-shift is perhaps not completely obvious, but this task can easily be accomplished when X is a separable Hilbert space. Indeed, in this case, for arbitrary $S, T \in L(X)$ the operator $T \otimes I$ and $I \otimes S$ on the Hilbert tensor product $X \otimes X$ commute, since

$$(T \otimes I)(I \otimes S) = T \otimes S = (I \otimes S)(T \otimes I);$$

see [62, Section 2.6] for a nice exposition of the theory of the Hilbert tensor product. Moreover, since $T \otimes I$ is unitarily equivalent to the Hilbert direct sum $\sum_{n=1}^{\infty} \oplus T$, it is easily seen that the failure of SVEP at a point λ extends from T to $T \otimes I$. In the same vein, it follows that $I \otimes S$ is a semi-shift whenever S is, since $I \otimes S$ is unitarily equivalent to $\sum_{n=1}^{\infty} \oplus S$. Note that, in the Hilbert spaces case , the semi-shifts are precisely pure isometries.

Thus neither SVEP nor localized SVEP is, in general, preserved under sums and products of commuting perturbations. We next will show that the SVEP is preserved under the special case of a Riesz commuting perturbation. Recall that an operator $R \in L(X)$ is said to be a *Riesz operator* if $\lambda I - R$ is a Fredholm operator for every $\lambda \in \mathbb{C} \setminus \{0\}$ and this is equivalent to saying that $\lambda I - R$ is Browder for all $\lambda \in \mathbb{C} \setminus \{0\}$. The spectrum $\sigma(R)$ of a Riesz operator is at most countable and has no nonzero cluster point. Furthermore, each nonzero element of the spectrum is an eigenvalue and the spectral projection associated with every $\lambda \neq 0$ is finite-dimensional, we refer to the book [1] for details. Examples of Riesz operators are quasi-nilpotent operators and compact operators, see [58]. It is well-known that if $R \in L(X)$ be a Riesz operator and Ω a spectral subset of $\sigma(R)$ such $0 \notin \Omega$, then the spectral projection P associated with Ω is finite dimensional.

The following recent result ([15]) shows that localized SVEP from an operator T is preserved under Riesz commuting perturbations.

Theorem 4.2. *Let X be a Banach space, $T, R \in L(X)$, where R is a Riesz operator such that $TR = RT$. If $\lambda \in \mathbf{C}$, then T has SVEP at λ if and only if $T - R$ has SVEP at λ. In particular, the SVEP is stable under Riesz commuting perturbations.*

Proof. Without loss of generality we may assume that $\lambda = 0$. Suppose T has not SVEP at 0. We show that $T - R$ has not SVEP at 0. Since T has not SVEP at 0, we have $\ker T \cap K(T) \neq \{0\}$, by Theorem 3.1, hence there exists a sequence of vectors $(x_i)_{i=0.1,\dots}$ of X such that $x_0 \neq 0$, $Tx_0 = 0$, $Tx_i = x_{i-1}$ $\quad (i \geq 1)$ and $\sup_{i \geq 1} \|x_i\|^{1/i} < \infty$. Let $K := \sup_{i \geq 1} \|x_i\|^{1/i}$ and fix an ε, $0 < \varepsilon < \frac{1}{2K}$. Let $\Omega := \{\lambda \in \sigma(R) : |\lambda| \geq \varepsilon\}$ and denote by P the spectral projection associated with Ω. Then P is finite dimensional, and if $X_2 := P(X)$ and $X_1 := \ker P$, then $X = X_1 \oplus X_2$. According the spectral decomposition theorem we have $R(X_j) \subset X_j \quad (j = 1, 2)$,

$$\sigma(R|X_1) \subset \{\lambda : |\lambda| < \varepsilon\} \quad \text{and} \quad \sigma(R|X_2) \subset \{\lambda : |\lambda| \geq \varepsilon\}.$$

Since $TR = RT$, we also have $T(X_j) \subset X_j \quad (j = 1, 2)$. Clearly,

$$TPx_0 = PTx_0 = 0,$$

and

$$TPx_i = PTx_i = Px_{i-1} \quad (i \geq 1).$$

We claim that $Px_i = 0$ for all i. To see this, suppose that $Px_i \neq 0$ for some $i \geq 0$. Since $TPx_{i+1} = Px_i \neq 0$ we then deduce that $Px_{i+1} \neq 0$, and by induction it then follows that $Px_n \neq 0$ for all $n \geq i$.
Let $k \geq 1$ be the smallest integer for which $Px_k \neq 0$. We have

$$TPx_k = Px_{k-1} = 0.$$

For all $n \geq k$ we also have

$$\begin{aligned} T^{n-k}Px_n &= T^{n-k-1}(TPx_n) = T^{n-k-1}Px_{n-1} = ... \\ &= TPx_{k+1} = Px_k \neq 0, \end{aligned}$$

hence $Px_n \notin \ker (T|X_2)^{n-k}$, for all $n \geq k$. Furthermore,

$$T^{n-k+1}Px_n = TT^{n-k}Px_n = TPx_k = Px_{k-1} = 0,$$

so $Px_n \in \ker (T|X_2)^{n-k+1}$. This implies that $T|X_2$ has infinite ascent, which is impossible, since $\dim X_2 < \infty$. Therefore, $Px_i = 0$, and hence $x_i \in \ker\, P = X_1$, for all $i \geq 0$.

Let us consider the restriction $R_1 = R|X_1$. Clearly, $r(R_1) < \varepsilon$, so there exists j_0 such that $\|R_1^j\| \leq \varepsilon^j$ for all $j \geq j_0$.

Set $y_0 := \sum_{i=0}^{\infty} R^i x_i$ and similarly, for $k \geq 1$ let

$$y_k := \sum_{i=k}^{\infty} \binom{i}{k} R^{i-k} x_i.$$

This definition is correct, since

$$\begin{aligned} \sum_{i=k}^{\infty} \binom{i}{k} \|R^{i-k}x_i\| &\leq \sum_{i=k}^{\infty} 2^i \|R_1^{i-k}\| K^i \\ &\leq \sum_{i=k}^{j_0+k} 2^i K^i \|R_1^{i-k}\| + \sum_{i=j_0+k+1}^{\infty} 2^i K^i \varepsilon^{i-k} < \infty. \end{aligned}$$

Moreover, for $k \geq 2j_0$ we have

$$\begin{aligned} \|y_k\| &\leq \sum_{i=k}^{2k-1} 2^i K^i \|R_1^{i-k}\| + \sum_{i=2k}^{\infty} (2K)^i \varepsilon^{i-k} \\ &\leq k \max\{(2K)^k, (2K)^{2k-1} \|R_1\|^{k-1}\} + \frac{(2K)^{2k}\varepsilon^k}{1 - 2K\varepsilon}. \end{aligned}$$

Hence,

$$\begin{aligned} \|y_k\|^{1/k} &\leq k^{1/k} \Big(\max\{(2K)^k, (2K)^{2k-1}\|R_1\|^{k-1}\}\Big)^{1/k} + \Big(\frac{(2K)^{2k}\varepsilon^k}{1-2K\varepsilon}\Big)^{1/k} \\ &\leq k^{1/k} \max\{2K, (2K)^{\frac{2k-1}{k}} \|R_1\|^{\frac{k-1}{k}}\} + \frac{4K^2\varepsilon}{1-2K\varepsilon}. \end{aligned}$$

from which we obtain $\limsup_{k\to\infty} \|y_k\|^{1/k} < \infty$.

We also have

$$(T - R)y_0 = \sum_{i=1}^{\infty} R^i x_{i-1} - \sum_{i=0}^{\infty} R^{i+1} x_i = 0.$$

Now, for $k \geq 1$ we have

$$\begin{aligned} (T - R)y_k &= \sum_{i=k}^{\infty} \binom{i}{k} R^{i-k} x_{i-1} - \sum_{i=k}^{\infty} \binom{i}{k} R^{i-k+1} x_i \\ &= x_{k-1} + \sum_{i=k}^{\infty} R^{i-k+1} x_i \Big(\binom{i+1}{k} - \binom{i}{k}\Big) = y_{k-1}. \end{aligned}$$

It remains to show that not all of the y_k's are equal to zero. Suppose on the contrary that $y_k = 0 \quad (k \geq 0)$ and let $j_1 \geq j_0$. Then we have

$$\sum_{k=0}^{j_1} (-1)^k R^k y_k = \sum_{i=0}^{\infty} \alpha_i R^i x_i,$$

where, if we let $\nu := \min\{i, j_1\}$, we have

$$\alpha_i = \sum_{k=0}^{\nu}(-1)^k\binom{i}{k} \quad \text{for every } i = 0, 1, \ldots.$$

Clearly, $\alpha_0 = 1$. For $1 \le i \le j_1$ we obtain

$$\alpha_i = \sum_{k=0}^{i}(-1)^k\binom{i}{k} = 0.$$

For $i > j_1$ we have $|\alpha_i| \le 2^i$, so

$$0 = \sum_{k=0}^{j_1}(-1)^k R^k y_k = x_0 + \sum_{i=j_1+1}^{\infty} \alpha_i R^i x_i$$

and

$$\|x_0\| \le \sum_{i=j_1+1}^{\infty} 2^i \|R_1^i\| \|x_i\| \le \sum_{i=j_1+1}^{\infty} 2^i \varepsilon^i K^i = \frac{(2K\varepsilon)^{j_1+1}}{1 - 2K\varepsilon}.$$

Letting $j_1 \to \infty$ yields $\|x_0\| = 0$, a contradiction. Therefore, $\ker(T - R) \cap K(T - R) \neq \{0\}$, and this implies, again by Theorem 3.1, that $T - R$ does not have SVEP at 0.

By symmetry we then conclude that T has SVEP at 0 if and only if $T - R$ has SVEP at 0. ∎

Remark 4.3. Every Riesz operator is *meromorphic*, i.e., every nonzero $\lambda \in \sigma(T)$ is a pole of the resolvent of T. Meromorphic operators have the same structure of the spectrum as Riesz operators, i.e., every $0 \neq \lambda \in \sigma(T)$ is an eigenvalue, and the spectrum is at most countable and has no nonzero cluster point, see [58, §54]. A simple example shows that the result of Theorem 4.2 cannot be extended to meromorphic operators. Denote by L is the backward shift on $\ell_2(\mathbb{N})$ and let $\lambda_0 \notin \sigma(L) = \mathbf{D}$, $\mathbf{D}$ the closed unit disc. It is known that L does not have SVEP at 0. Since L has SVEP at λ_0, then $T := \lambda_0 I - L$ has SVEP at 0, while $T - \lambda_0 I = -L$, does not have SVEP at 0, and, obviously, $\lambda_0 I$ is meromorphic.

Remark 4.4. Denote by $\sigma_e(T)$ is the essential Fredholm spectrum of T, and let $r_e(T)$ denote the essential spectral radius of T. i.e.,

$$r_e(T) := \sup\{|\lambda| : \lambda \in \sigma_e(T)\}.$$

Obviously, in the case of a Riesz operator K we have $r_e(K) = 0$. A closer look at the proof of Theorem 4.2 reveals that the stability of localized SVEP also holds if we assume that $r_e(K)$ is small enough.

The result of Theorem 4.2 implies that the localized SVEP is stable under quasi-nilpotent commuting perturbations. We now address the question of the extent to which SVEP at a point is stable under quasi-nilpotent equivalence. Before we need to give some definitions.

Definition 4.5. *The operator $A \in L(X, Y)$ between the Banach spaces X and Y is a* quasi-affinity *if it has a trivial kernel and dense range. We say that $T \in L(X)$ is a* quasi-affine transform *of $S \in L(Y)$, and we write $T \prec S$, if there is a quasi-affinity $A \in L(X, Y)$ that intertwines T and S, i.e. $SA = AT$. If there exists two quasi-affinities $A \in L(X, Y)$, $B \in L(X, Y)$ for which $SA = AT$ and $BS = TB$ then we say that S and T are* quasi-similar.

The *commutator* of two operators $S, T \in L(X)$ is the operator $C(S, T)$ on $L(X)$ defined by

$$C(S, T)(A) := SA - AT \quad \text{for all } A \in L(X).$$

By induction it is easily to show the binomial identity

$$C(S,T)^n(A) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^k S^{n-k} A T^k. \tag{13}$$

Obviously, $C(\lambda I - S, \lambda I - T)^n(A) = (-1)^n C(S,T)^n(A)$ for all $\lambda \in \mathbb{C}$, from which we obtain

$$\begin{aligned} C(S,T)^n(A) &= (-1)^n C(\lambda I - S, \lambda I - T)^n(A) \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (\lambda I - S)^{n-k} A (\lambda I - T)^k \end{aligned}$$

for all $A \in L(X)$, $n \in \mathbb{N}$. The equality (13) also entails that

$$C(S,T)^n(A)x = S^n A x \quad \text{for all } x \in \ker T. \tag{14}$$

Definition 4.6. *Given the operators $T \in L(X)$ and $S \in L(Y)$, we say that the pair (S,T) is* asymptotically intertwined *by the operator $A \in L(X,Y)$ if $\|C(T,S)(A)\|^{1/n} \to 0$ as $n \to \infty$ The operators $S \in L(X)$ an $T \in L(X)$ are said to be* quasi-nilpotent equivalent *if (S,T) and (T,S) are asymptotically intertwined by the identity operator I on X.*

Evidently, the notion of asymptotically intertwined pairs is a generalization of the intertwining condition $C(S,T)(A) = 0$ which appears in the definition of $T \prec S$. This notion is also a generalization of the higher order intertwining condition:

$$C(S,T)^n(A) = 0 \quad \text{for some } n \in \mathbb{N}.$$

Lemma 4.7. *[9] Let $T \in L(X)$, $S \in L(Y)$ and suppose that for some injective map $A \in L(X,Y)$ there exists an integer $n \in \mathbb{N}$ for which $C(S,T)^n(A) = 0$. If S has SVEP at λ_0 then T has SVEP at λ_0. In particular, if $T \in L(X)$ and $S \in L(Y)$ are intertwined by an injective map $A \in L(X,Y)$ then localized SVEP carries over from S to T.*

The following example shows that converse of Lemma 4.7 does not hold, i.e. if $T \prec S$ the SVEP from T may be not transmitted to S.

Example 4.8. Let C denote the *Cesàro matrix*,i.e. C is a lower triangular matrix such that the nonzero entries of the n-th row are n^{-1} ($n \in \mathbf{N}$)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & 0 & \cdots \\ 1/4 & 1/4 & 1/4 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Let $1 < p < \infty$ and consider the matrix C as an operator C_p acting on ℓ_p. Let q be such that $1/p + 1/q = 1$. In [83] it has been proved that $\sigma(C_p)$ is the closed disc Γ_q, where

$$\Gamma_q := \{\lambda \in \mathbb{C} : |\lambda - q/2| \leq q/2\}.$$

Moreover, it has been proved in [51] that for each $\mu \in \operatorname{int}\Gamma_q$ the operator $\mu I - C_p$ is an injective Fredholm operator with $\beta(C_p) = 1$, . Consequently, every $\mu \in \operatorname{int}\Gamma_q$ belongs to the surjectivity spectrum $\sigma_s(C_p)$.

Let $C_p{}^* \in L(\ell_q)$ denote the conjugate operator of C_p. Obviously, $\sigma_s(C_p)$ clusters at every $\mu \in \operatorname{int}\Gamma_q$ and since $\mu I - C_p$ is Fredholm it then follows that $C_p{}^*$ does not have SVEP at these points μ, by Theorem 3.19. Every operator has SVEP at the boundary of the spectrum, and since $\sigma(C_p^*) = \sigma(C_p) = \Gamma_q$ it then follows that $C_p{}^*$ has SVEP at λ precisely when $\lambda \notin \operatorname{int}\Gamma_q$. Choose $1 < p' < p < \infty$ and let q' be such that $1/p' + 1/q' = 1$. Then $1 < q < q' < \infty$. If we denote by $A : \ell_q \to \ell_{q'}$ the natural inclusion then we have $C_{p'}^* A = A C_p^*$ and clearly A is an injective operator with dense range, hence $C_p^* \prec C_{p'}^*$. As noted before the operator C_p^* has SVEP at every point outside of Γ_q, in particular C_p^* has SVEP at the points $\lambda \in \Gamma_{q'} \setminus \Gamma_q$, while $C_{p'}^*$ fails SVEP at the points $\lambda \in \Gamma_{q'} \setminus \Gamma_q$ which do not belong to the boundary of $\Gamma_{q'}$.

The following permanence results require a rather technical work, the reader can be find the proof of these results in [68, Chapter 3].

Theorem 4.9. *Quasi-nilpotent equivalence preserves SVEP. Moreover, quasi-nilpotent equivalent operators have the same local spectra, the same surjectivity spectrum, the same approximate point spectrum, and the same spectrum. Furthermore, if T and S are quasi-nilpotent equivalent then the identity $\mathcal{X}_T(\Omega) = \mathcal{X}_S(\Omega)$ holds for every closed subset Ω of $\mathbb{C}$.*

Theorem 4.9 then implies that the identity $X_T(\Omega) = X_S(\Omega)$ holds for every closed subset Ω of $\mathbb{C}$. Moreover, Since by Theorem 1.9 an operator $T \in L(X)$ has SVEP precisely when $X_T(\emptyset) = \{0\}$, and since quasi-nilpotent equivalence preserves the analytic spectral subspaces, it is clear that SVEP is stable under quasi-nilpotent equivalence. If there exists an integer $n \in \mathbb{N}$ for which $C(S,T)^n(I) = C(T,S)^n(I) = 0$, then the operators S and T are said to be *nilpotent equivalent*. For $S, T \in L(X)$ with $ST = TS$, it is easily seen that

$$C(S,T)^n(I) = (S-T)^n \quad \text{for all } n \in \mathbb{N}.$$

Thus, in this case, S and T are quasi-nilpotent equivalent precisely when $S - T$ is quasi-nilpotent, while S and T are nilpotent equivalent if and only if $S - T$ is nilpotent.

Theorem 4.10. *[16] Suppose that the operators $S, T \in L(X)$ are nilpotent equivalent, and let $\lambda \in \mathbb{C}$. Then T has SVEP at λ precisely when S does. In particular, if T has SVEP at λ, and if $N \in L(X)$ is nilpotent and satisfies $TN = NT$, then also $T + N$ has SVEP at λ.*

We know that localized SVEP is stable under commuting quasi-nilpotent perturbations. A natural question is if the SVEP at a point is preserved under under quasi-nilpotent equivalence. Although we do not know the answer to this question in general, we can handle certain important special cases.

Nilpotent operators are special cases of algebraic operators. Recall that an operator $K \in L(X)$ is said to be *algebraic* if there exists a non-trivial complex polynomial h such that $h(K) = 0$. In addition to nilpotent operators, examples of algebraic operators are idempotent operators and operators for which some power has finite-dimensional range. Note that if K is algebraic, by the classical spectral mapping theorem we have $h(\sigma(K)) = \sigma(h(K)) = \{0\}$, so the spectrum $\sigma(K)$ is finite.

If $T \in L(X)$ has *SVEP* at a point λ, then it may be tempting to conjecture that $T + K$ has *SVEP* at λ for every algebraic operator $K \in L(X)$ that commutes with T. However, this cannot be true in general. Indeed, in the example given in Remark 4.3, the operator $K := -\lambda_0 I$ is obviously algebraic, T has SVEP at 0 while $T + K$ does not have SVEP at 0. Nevertheless, we obtain the following result.

Theorem 4.11. *[16] Let $T, K \in L(X)$ be commuting operators, suppose that K is algebraic, and let h be a non-zero polynomial for which $h(K) = 0$. If T has SVEP at each of the zeros of h, then $T - K$ has SVEP at 0. In particular, if T has SVEP, then so does $T + K$.*

The case of commuting quasi-nilpotent equivalence seems to be more complicated. In the next theorem we assume that $H_0(\lambda I - T) \cap X_T(\emptyset) = \{0\}$. This condition, as it has been Theorem 3.8, is stronger than T has SVEP at λ.

Theorem 4.12. *[16] Suppose that $T \in L(X)$ satisfies $H_0(\lambda I - T) \cap X_T(\emptyset) = \{0\}$ for some $\lambda \in \mathbb{C}$, and let $S \in L(X)$ be quasi-nilpotent equivalent to T. Then S has SVEP at λ.*

The SVEP at a point is preserved under quasi-nilpotent equivalence if we assume that $\lambda I - T$ either admits a generalized Kato decomposition or is quasi-Fredholm.

Corollary 4.13. *Let $S, T \in L(X)$ be quasi-nilpotent equivalent operators, let $\lambda \in \mathbb{C}$, and suppose that $\lambda I - T$ either admits a generalized Kato decomposition or is quasi-Fredholm. If T satisfies* SVEP *at λ, then so does S.*

We finally address the permanence of localized SVEP for the adjoint T^* of an operator $T \in L(X)$. The condition $H_0(\lambda I-T)+K(\lambda I-T) = X$ may be thought of as being dual to the condition $H_0(\lambda I-T)\cap K(\lambda I-T) = \{0\}$, and entails the SVEP for T^* at λ, by Theorem 3.8. These observations concerning the localized SVEP will be improved in the following result.

Theorem 4.14. *[16] For every pair of quasi-nilpotent equivalent operators $S, T \in L(X)$ and arbitrary $\lambda \in \mathbb{C}$, the following assertion hold:*

(i) *if $K(\lambda I - T) + H_0(\lambda I - T)$ is norm dense in X, then S^* has SVEP at the point λ.*

(ii) *if $H_0(\lambda I - T^*) + K(\lambda I - T^*)$ is weak-*-dense in X^*, then S has SVEP at the point λ.*

5. Weyl and Browder spectra under perturbations

In this section we prove several perturbation result concerning the spectra relative to some important classes of operators in Fredholm theory, some of them have been already introduced in the previous sections. Recall that a bounded operator $T \in L(X)$ is said to be a *Weyl operator*, $T \in W(X)$, if T is a Fredholm operator having index 0. The classes of *upper semi-Weyl* and *lower semi-Weyl* operators are defined, respectively:

$$W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind} T \leq 0\},$$

$$W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind} T \geq 0\}.$$

Clearly, $W(X) = W_+(X) \cap W_-(X)$. The *Weyl spectrum* is defined as

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\},$$

the *upper semi-Weyl spectrum* is defined as

$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\},$$

and the *lower semi-Weyl spectrum* is defined as

$$\sigma_{\mathrm{lw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_-(X)\}.$$

By duality we have $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T^*)$ and

$$\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{lw}}(T^*) \quad \text{and} \quad \sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{uw}}(T^*).$$

Clearly, every Browder (respectively, upper semi-Browder, lower semi-Browder) operator $T \in L(X)$ is Weyl (respectively, upper semi-Weyl, lower semi-Weyl), by Theorem 2.7, so $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$, $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$ and $\sigma_{\mathrm{lw}}(T) \subseteq \sigma_{\mathrm{lb}}(T)$. Moreover, by duality, $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{b}}(T^*)$ and

$$\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{lb}}(T^*) \quad \text{and} \quad \sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ub}}(T^*).$$

We now turn to the stability of semi-Browder spectra, or more generally, of the essentially semi-regular spectrum under commuting Riesz perturbations. In the next lemma we collect some other basic facts about Riesz operators that will be used in the rest of the chapter.

Lemma 5.1. *Let $R \in L(X)$ be a Riesz operator. Then we have*

(i) *R^* is a Riesz operator.*

(ii)*If M is a closed R-invariant subspace of R then both the restriction $R|M$ and the operator $\tilde{R} : X/M \to X/M$ induced by R are Riesz.*

Semi-Browder operators are stable under commuting Riesz perturbations:

Theorem 5.2. *[82] Let $T \in L(X)$ and R a Riesz operator such that $TR = RT$. Then we have:*

(i) $T \in B_+(X) \Leftrightarrow T + R \in B_+(X)$.

(ii) $T \in B_-(X) \Leftrightarrow T + R \in B_+(X)$.

(i) $T \in B(X) \Leftrightarrow T + R \in B(X)$.

Proof. (i) Let T be upper semi-Browder, so $p(T) < \infty$ and this is equivalent to saying that T has SVEP at 0, by Theorem 3.18. By Theorem 4.2 then $T + R$ has SVEP at 0, and since $T + R$ is upper semi-Fredholm it then follows that $p(T+R) < \infty$, so $T+R$ is upper semi-Browder. The converse implication follows by symmetry.

(ii) The proof is analogous to that of part (i). Let T be lower semi-Browder, so $q(T) < \infty$ and this is equivalent to saying that T^* has SVEP at 0, by Theorem 3.19. The dual of a Riesz operator is also Riesz. By Theorem 4.2 it then follows that $T^* + R^*$ has SVEP at 0, and since $T + R$ is lower semi-Fredholm it then follows that $q(T + R) < \infty$, so $T + R$ is lower semi-Browder.

(iii) Clear.

Corollary 5.3. *The Browder spectra $\sigma_{\mathrm{ub}}(T)$, $\sigma_{\mathrm{lb}}(T)$, and $\sigma_{\mathrm{b}}(T)$ are stable under commuting Riesz perturbations.*

In the particular case of bounded below, or surjective, operators we can say something more:

Theorem 5.4. *Suppose that $T \in L(X)$ and $R \in L(X)$ a Riesz operator commuting with T. Then*

(i) *If T is bounded below then $T + K \in B_+(X)$. Moreover,*

$$T(\ker (T + R)^n) = \ker (T + R)^n \quad \textit{for all } n \in \mathbb{N}.$$

(ii) *If T is onto then $T + K \in B_(X)$. Moreover,*

$$T^{-1}((T + R)^n(X)) = (T + R)^n(X) \quad \textit{for all } n \in \mathbb{N}.$$

Proof. (i) The first assertion is clear by Theorem 5.2, since $T \in B_+(X)$.

Define $S := T+R$. We have $S^n \in \Phi_+(X)$ for every $n \in \mathbb{N}$, so $\ker S^n$ is finite dimensional. If $x \in \ker S^n$ then $Tx \in \ker S^n$, since $(T + R)^n Tx = T(T + R)^n x = 0$, hence $\ker S^n$ is T-invariant. Furthermore, the restriction of T to $\ker S^n$ is injective, since T is bounded below, so T maps $\ker S^n$ onto itself.

(ii) The first assertion is clear by Theorem 5.2, since $T \in B_-(X)$.

Let $S := T + R$. Then $S^n \in \Phi_-(X)$ for every $n \in \mathbb{N}$, so $\operatorname{codim} S^n(X) = \dim X/S^n(X) < \infty$. Consider the map $\hat{T} : X/S^n(X) \to X/S^n(X)$ induced by T , defined by $\hat{T}\hat{x} := \hat{T}x$ for all $\hat{x} := x + S^n(X)$. Since T is onto, for every $y \in X$ there exists an element $z \in X$ such that $y = Tz$, and therefore, $\hat{y} = \hat{T}\hat{z}$. Hence $\hat{T}$ is onto. Since $X/S^n(X)$ is finite dimensional it then follows that $\hat{T}$ is also injective, and this implies that $Tx \in S^n(X)$ if and only if $x \in S^n(X)$. Consequently, $T^{-1}(S^n(X)) = S^n(X)$. ∎

Lemma 5.5. *Suppose that $T \in L(X)$ is essentially semi-regular. Then the operator $\tilde{T} : X/T^\infty(X) \to X/T^\infty(X)$ is upper semi-Browder.*

Proof. Let (M, N) be the corresponding Kato decomposition for which $X = M \oplus N$, $T|M$ is semi-regular, $T|N$ nilpotent with $\dim N < \infty$. Clearly, $T^\infty(X) = (T|M)^\infty(M) \subseteq M$. Moreover, $T^\infty(X)$ is closed and $T(T^\infty(X)) = T^\infty(X)$. Let $k \geq 1$ and $x = x_1 \oplus x_2$ satisfies $T^k x \in T^\infty(X)$. Then $(T|M)x_2 \in T^\infty(X)$, thus $x \in N + T^\infty(X)$ and $\dim \ker T^k \leq \dim N$. Consequently, $\mathcal{N}^\infty(\tilde{T}) \leq \dim N < \infty$. Let $\pi : X \to X/T^\infty(X)$ be the canonical projection. As $T^\infty(X) \subseteq T(X)$ and the range of $\tilde{T}$ is the set $\{Tx + T^\infty(X), x \in X\} = \pi(T(X))$, the range of $\tilde{T}$ is closed, hence $\tilde{T}$ is upper semi-Browder. ∎

Also the class of essentially semi-regular operators is stable under Riesz commuting perturbations:

Theorem 5.6. *[65] Suppose that $T, K \in L(X)$ commutes, T is essentially semi-regular and K is a Riesz operator. Then $T + K$ is essentially semi-regular.*

Proof. The subspace $T^\infty(X)$ is closed. Set $\hat{T} := T|T^\infty(X)$ and let $\tilde{T} : X/T^\infty(X) \to X/T^\infty(X)$ be the operator induced by T. Observe that $TS = ST$ entails that $S(T^\infty(X)) \subseteq T^\infty(X)$, so $T^\infty(X)$ is both T-invariant and S-invariant. Since T has topological uniform descent then Theorem 2.17 implies that $\hat{T}$ is onto. The restriction $\hat{S} := S|T^\infty(X)$ is Riesz, and since $\hat{T}\hat{S} = \hat{S}\hat{T}$, Theorem 5.4 entails that $\hat{T} + \hat{S}$ is lower semi-Browder. Now, let $\tilde{T}$ and $\tilde{S}$ denote the induced mappings on $X/T^\infty(X)$, by T and S respectively. From Lemma 5.5 we know that $\tilde{T}$ is upper semi-Browder, $\tilde{S}$ is Riesz and $\tilde{S}\tilde{T} = \tilde{T}\tilde{S}$, thus, by Theorem 5.2, $\tilde{T} + \tilde{S}$ is upper semi-Browder. By Theorem 2.8, applied to $T + S$, we then conclude that $T + S$ is essentially semi-regular. ∎

Denote by

$$\sigma_{\rm es}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not essentially semi-regular}\}$$

the *essentially semi-regular spectrum*. Obviously, $\sigma_{\rm es}(T)$ is a subset of the semi-regular spectrum $\sigma_{\rm se}(T)$.

Theorem 5.7. *[78, §21], [81] Let $T \in L(X)$.*

(i) *$\sigma_{\rm es}(T)$ is a non-empty compact subset. In particular, $\sigma_{\rm es}(T)$ contains the boundary of the essential spectrum $\sigma_{\rm e}(T)$.*

(ii) *If $K \in L(X)$ is finite-dimensional (not necessarily commuting) or a commuting Riesz operator then $\sigma_{\rm es}(T) = \sigma_{\rm es}(T + K)$.*

(iii) *$\sigma_{\rm es}(T) = \bigcap \sigma_{\rm se}(T + K)$, where the intersection is taken over all Riesz operators commuting with T, or equivalently the intersection is taken over all finite rank operators commuting with T.*

Essentially semi-regular operators having finite ascent, or finite descent, are also stable under Riesz commuting perturbations:

Theorem 5.8. *[19] Let $T \in L(X)$ is essentially semi-regular and $K \in L(X)$ a Riesz operator commuting with T. Then*

(i) *T has finite ascent if and only if $T + K$ has finite ascent.*

(ii) *T has finite descent if and only if $T + K$ has finite descent.*

Proof. (i) Suppose that T has finite ascent and K a Riesz operator commuting with T. We know that the condition $p(T) < \infty$ entails that T has SVEP at 0. Hence $T + K$ has SVEP at 0, by Theorem 4.2. But $T + K$ is essentially semi-regular, by Theorem 5.6, in particular has topological uniform descent. The SVEP of $T + K$ at 0 is then equivalent to saying that $p(T + K) < \infty$, by Theorem 3.18. The converse may be obtained by symmetry from the equality $p(T) = p((T + K) - K) = p(T + K)$, since $T + K$ commutes with K.

The proof of part (ii) is analogous: if T has finite descent and K is Riesz, then T^* has SVEP at 0, hence, by Theorem 4.2, $T^* + K^*$ has SVEP at 0, since K^* is a Riesz operator, and K^* commutes with T^*. Now, $T + K$ is essentially semi-regular, always by Theorem 5.6, and hence $T^* + K^*$ is essentially semi-regular, see [1, Corollary 1.49], in particular has topological uniform descent. The SVEP of $T^* + K^*$ at 0 then implies, by Theorem 3.18, that $q(T + K) < \infty$. The converse may be obtained always by symmetry. ∎

Theorem 5.9. *[1, Theorem 3.39] For a bounded operator T on a Banach space X, the following assertions are equivalent:*

(i) *T is a Weyl operator;*

(ii) *There exist $K \in \mathcal{F}(X)$ and an invertible operator $S \in L(X)$ such that $T = S + K$ is invertible;*

(iii) *There exist $K \in \mathcal{K}(X)$ and an invertible operator $S \in L(X)$ such that $T = S + K$ is invertible.*

By means of a simple modification of the proof of Theorem 5.9 we easily obtain the following characterizations of upper and lower Weyl operators::

Theorem 5.10. *[1, p. 135] Let $T \in L(X)$. Then we have*

(i) *$T \in \mathcal{W}_+(X)$ if and only if there exist $K \in \mathcal{K}(X)$ and a bounded below operator S such that $T = S + K$.*

(ii) *$T \in \mathcal{W}_-(X)$ if and only if there exist $K \in \mathcal{K}(X)$ and a surjective operator S such that $T = S + K$.*

As a simple consequence of Theorem 5.9 and Theorem 5.10, the Weyl spectra may be characterized in terms of commuting perturbations as follows.

Corollary 5.11. *Let $T \in L(X)$. Then we have*

$$\sigma_{\mathrm{uw}}(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma_{\mathrm{ap}}(T+K), \quad \sigma_{\mathrm{lw}}(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma_{\mathrm{s}}(T+K), \tag{15}$$

and

$$\sigma_{\mathrm{w}}(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma(T+K). \tag{16}$$

The next result shows that the Browder spectra may be obtained by adding to Weyl spectra the cluster points of parts of the spectrum.

Theorem 5.12. *[1, Theorem 3.65] For a bounded operator $T \in L(X)$ the following statements hold:*

(i) $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{uw}}(T) \cup \operatorname{acc} \sigma_{\mathrm{ap}}(T)$.

(ii) $\sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{lw}}(T) \cup \operatorname{acc} \sigma_{\mathrm{s}}(T)$.

(iii) $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) \cup \operatorname{acc} \sigma(T)$.

If either T or T^* has SVEP we can say more:

Theorem 5.13. *[1, Theorem 3.66] Suppose that $T \in L(X)$.*

(i) *If T has SVEP then $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{lb}}(T)$.*

(ii) *If T^* has SVEP then $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{ub}}(T)$.*

(iii) *If either T or T^* has the SVEP we have*

$$\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T) \quad \text{and} \quad \sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{lb}}(T).$$

We now give a characterization of semi-Browder operators by means of the SVEP.

Theorem 5.14. *[1, Theorem 3.45] For an operator $T \in L(X)$, X a Banach space, the following statements are equivalent:*

(i) *$T \in \mathcal{B}_+(X)$;*

(ii) *There exist a $K \in \mathcal{F}(X)$ and a bounded below operator $S \in L(X)$ such that $TK = KT$ and $T = S + K$ is bounded below ;*

(iii) *There exists a $K \in \mathcal{K}(X)$ and a bounded below operator $S \in L(X)$ such that $TK = KT$ and $T = S + K$;*

(iv) *T is essentially semi-regular and T has the SVEP at 0.*

The next result is dual to that given in Theorem 5.14.

Theorem 5.15. *[1, Theorem 3.45] Let $T \in L(X)$, X a Banach space. Then the following properties are equivalent:*

(i) *$T \in \mathcal{B}_-(X)$;*

(ii) *There exist a $K \in \mathcal{F}(X)$ and a surjective operator S such that $TK = KT$ and $T = S + K$;*

(iii) *There exist a $K \in \mathcal{K}(X)$ and a surjective operator S such that $TK = KT$ and $T = S + K$;*

(iv) *T is essentially semi-regular and T^* has the SVEP at 0.*

Combining Theorem 5.14 and Theorem 5.15 we readily obtain the following characterizations of Browder operators.

Theorem 5.16. *Let $T \in L(X)$, X a Banach space. Then the following properties are equivalent:*

(i) $T \in \mathcal{B}(X)$;

(ii) *There exist $K \in \mathcal{F}(X)$ and an invertible operator S such that $TK = KT$ and $T = S + K$;*

(iii) *There exist $K \in \mathcal{K}(X)$ and an invertible operator S such that $TK = KT$ and $T = S + K$;*

(iv) *T is essentially semi-regular, both T and T^* have SVEP at 0.*

From Theorem 5.14, Theorem 5.15 and Theorem 5.16 we easily obtain:

Corollary 5.17. *Let $T \in L(X)$. Then we have*

$$\sigma_{\rm ub}(T) = \bigcap_{K\in\mathcal{K}(X),KT=TK} \sigma_{\rm ap}(T+K), \quad \sigma_{\rm lb}(T) = \bigcap_{K\in\mathcal{K}(X),KT=TK} \sigma_{\rm s}(T+K), \tag{17}$$

and

$$\sigma_{\rm b}(T) = \bigcap_{K\in\mathcal{K}(X),KT=TK} \sigma(T+K). \tag{18}$$

It is evident that the concepts of Weyl operators and Browder operators may be extended to B-Fredholm theory. Precisely, if $T^n(X)$ is closed and $T_{[n]} : T|T^n(X)$ is upper semi-Weyl (respectively, lower semi-Weyl, Weyl), then is said to be *upper semi B-Weyl*, (respectively, *lower semi B-Weyl, B-Weyl*). By $\sigma_{\rm ubw}(T)$, $\sigma_{\rm lbw}(T)$, $\sigma_{\rm bw}(T)$, we denote the upper semi B-Weyl spectrum, the lower semi B-Weyl spectrum and the B-Weyl spectrum, respectively. Analogously, $T \in L(X)$ is said to be *upper semi B-Browder*, (respectively, *lower semi B-Browder, B-Browder*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is upper semi-Browder (respectively, lower semi-Browder, Browder.) By $\sigma_{\rm ubb}(T)$, $\sigma_{\rm lbb}(T)$, $\sigma_{\rm bb}(T)$, we denote the upper semi B-Browder spectrum, the lower semi B-Browder spectrum and the B-Browder spectrum, respectively.

Every bounded below operator $T \in L(X)$ is upper semi-Browder, while every surjective operator $T \in L(X)$ is lower semi Browder, so every left Drazin invertible operator is upper semi B-Browder, while every right Drazin invertible operator is lower semi B-Browder. Actually, we have the following equivalences:

Theorem 5.18. *[5] If $T \in L(X)$ then the following equivalences hold:*

(i) *T is upper semi B-Browder $\Leftrightarrow$ T is left Drazin invertible.*

(ii) *T is lower semi B-Browder $\Leftrightarrow$ T if T is right Drazin invertible.*

Consequently, T is B-Browder if and only if T is Drazin invertible.

The *left Drazin spectrum* is defined as

$$\sigma_{\rm ld}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\},$$

the *right Drazin spectrum* is defined as

$$\sigma_{\rm rd}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not right Drazin invertible}\},$$

and the *Drazin spectrum* is defined as

$$\sigma_{\rm d}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\}.$$

Obviously, $\sigma_{\rm d}(T) = \sigma_{\rm ld}(T) \cup \sigma_{\rm rd}(T)$.

The relationship between the B-Browder spectra and the B-Weyl spectra is similar to that observed for the Browder spectra and Weyl spectra, established in Theorem 5.12:

Theorem 5.19. *[5] If $T \in L(X)$ then the following equalities hold:*

(i) $\sigma_{\rm ld}(T) = \sigma_{\rm ubw}(T) \cup acc\, \sigma_{\rm ap}(T)$.

(ii) $\sigma_{\rm rd}(T) = \sigma_{\rm lbw}(T) \cup acc\, \sigma_{\rm s}(T)$.

(iii) $\sigma_{\rm d}(T) = \sigma_{\rm bw}(T) \cup acc\, \sigma(T)$.

The following result shows that many of the spectra considered before coincide whenever T or T^* has SVEP.

Theorem 5.20. *Suppose that $T \in L(X)$. Then the following statements hold:*

(i) *If T has SVEP then*

$$\sigma_{\mathrm{lbw}}(T) = \sigma_{\mathrm{lbb}}(T) = \sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{bw}}(T). \tag{19}$$

(ii) *If T^* has SVEP then*

$$\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T). \tag{20}$$

(iii) *If both T and T^* have SVEP then*

$$\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{lbw}}(T) = \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T). \tag{21}$$

6. polaroid type operators

The concept of pole may be sectioned as follows:

Definition 6.1. *Let $T \in L(X)$, X a Banach space. If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\mathrm{ap}}(T)$ then λ is said to be a* left pole. *A left pole λ is said to have finite rank if $\alpha(\lambda I - T) < \infty$. If $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\mathrm{s}}(T)$ then λ is said to be a* right pole. *A right pole λ is said to have finite rank if $\beta(\lambda I - T) < \infty$.*

Clearly, λ is pole of the resolvent if and only λ if is both a right pole and a left pole.

In the sequel $\mathcal{H}(\Omega, Y)$, Y any Banach space, denotes the Fréchet space of all analytic functions from the open set $\Omega \subseteq \mathbb{C}$ to Y. We have proved in Theorem 1.32 that if $\lambda \in \operatorname{iso} \sigma_{\mathrm{ap}}(T)$ then $H_0(\lambda I - T)$ is closed. If λ is a left pole we can say more:

Theorem 6.2. *[18] Let $T \in LX)$, X a Banach space.*

(ii) *If λ is a left pole of $T \in L(X)$ then λ is an isolated point of $\sigma_{\mathrm{ap}}(T)$ and there exists $\nu \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = \ker (\lambda I - T)^\nu.$$

Moreover, λ is a left pole of finite rank then $H_0(\lambda I - T)$ is finite-dimensional.

(ii) *If λ is a right pole of $T \in L(X)$ then λ is an isolated point of $\sigma_{\mathrm{s}}(T)$, and there exists $\nu \in \mathbb{N}$ such that*

$$K(\lambda I - T) = (\lambda I - T)^\nu(X).$$

Moreover, λ is a right pole of finite rank if and only if $K(\lambda I - T)$ has finite codimension.

Remark 6.3. It should be noted that a left pole, as well as a right pole, need not to be an isolated point of $\sigma(T)$. For instance, let $R \in \ell^2(\mathbb{N})$ be the classical unilateral right shift and

$$U(x_1, x_2, \dots) := (0, x_2, x_3, \cdots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Define $T := R \oplus U$. Then $\sigma(T) = \mathbf{D}$, $\mathbf{D}$ the closed unit disc of $\mathbb{C}$. Moreover, $\sigma_{\mathrm{ap}}(T) = \Gamma \cup \{0\}$, Γ the unit circle, and T is upper semi-Browder, in particular left Drazin invertible. Hence 0 is a left pole (of finite rank, since $\alpha(T) = 1$) but $0 \notin \operatorname{iso} \sigma(T) = \emptyset$. Note that 0 is a right pole of the dual T^*, but is not an isolated point of $\sigma(T^*) = \sigma(T) = \mathbf{D}$.

In the case of Hilbert space operators we have much more. .

Theorem 6.4. *[18] Let $T \in L(H)$, H a Hilbert space.*

(i) *$\lambda \in \sigma_{\mathrm{ap}}(T)$ is a left pole if and only if there exist two T-invariant closed subspaces M, N such that $H = M \oplus N$, $\lambda I - T|M$ is bounded below, $\lambda I - T|N$ is nilpotent. In this case $N = H_0(\lambda I - T)$.*

(ii) *If $\lambda \in \sigma_{\mathrm{s}}(T)$ then λ is a right pole if and only if there exist two T-invariant closed subspaces M, N such that $H = M \oplus N$, $\lambda I - T|M$ is onto, $\lambda I - T|N$ is nilpotent. In this case $M = K(\lambda I - T)$.*

Corollary 6.5. *If $T \in L(H)$, H a Hilbert space, then λ is a left pole of finite rank if and only if the exists two closed T-invariant subspaces M, N such that $X = M \oplus N$, N is finite-dimensional, $\lambda I - T|M$ is bounded below and $\lambda I - T|N$ is nilpotent. Analogously, λ is a right pole of finite rank if and only if there exist two closed T-invariant subspaces M, N such that $X = M \oplus N$, M is finite-codimensional, $\lambda I - T|M$ is onto and $\lambda I - T|N$ is nilpotent*

We now introduce some classes of operators which have a very nice structure.

Definition 6.6. *A bounded operator $T \in L(X)$ is said to be* left polaroid *if every $\lambda \in \mathrm{iso}\,\sigma_{\mathrm{ap}}(T)$ is a left pole of the resolvent of T. $T \in L(X)$ is said to be* right polaroid *if every $\lambda \in \mathrm{iso}\,\sigma_{\mathrm{s}}(T)$ is a right pole of the resolvent of T. $T \in L(X)$ is said to be* polaroid *if every $\lambda \in \mathrm{iso}\,\sigma(T)$ is a pole of the resolvent of T. A bounded operator $T \in L(X)$ is said to be a-*polaroid *if every $\lambda \in \mathrm{iso}\,\sigma_{\mathrm{ap}}(T)$ is a pole of the resolvent of T.*

The concept of left and right polaroid are dual each other:

Theorem 6.7. *[4] If $T \in L(X)$ then λ is a left pole (respectively, right pole) of the resolvent of T if and only if λ is a right pole (respectively, left pole) of the resolvent of T^*. Consequently, T is left polaroid (respectively, right polaroid, polaroid) if and only if T^* is right polaroid (respectively, left polaroid, polaroid).*

Proof. The proof is immediate from Theorem 2.45, taking into account that $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{s}}(T^*)$ and $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{ap}}(T^*)$. ∎

The condition of being polaroid, may be characterized by means of the quasi-nilpotent part as follows:

Theorem 6.8. *[9] If $T \in L(X)$ the following statements hold:*

(i) *T is polaroid if and only if there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathrm{iso}\,\sigma(T). \tag{22}$$

(ii) *If T is left polaroid then there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathrm{iso}\,\sigma_{\mathrm{ap}}(T). \tag{23}$$

The relationships between the polaroid conditions are established in the following theorem.

Theorem 6.9. *If $T \in L(X)$ the following implications hold:*

$$T \text{ } a\text{-polaroid} \Rightarrow T \text{ left polaroid} \Rightarrow T \text{ polaroid}$$

Furthermore, if T is right polaroid then T is polaroid.

Proof. The first implication is clear, since a pole is always a left pole. Assume that T is left polaroid and let $\lambda \in \mathrm{iso}\,\sigma(T)$. It is known that the boundary of the spectrum is contained in $\sigma_{\mathrm{a}}(T)$, in particular every isolated point of $\sigma(T)$, thus $\lambda \in \mathrm{iso}\,\sigma_{\mathrm{a}}(T)$ and hence λ is a left pole of the resolvent of T. By part (ii) of Theorem 6.8 then there exists a natural $\nu := \nu(\lambda I - T) \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^{\nu}$. But λ is isolated in $\sigma(T)$ so, by part (i) of 6.8, λ is is a pole of the resolvent, i.e. T is polaroid.

To show the last assertion suppose that T is right polaroid. By Theorem 6.7 then T^* is left polaroid and hence, by the first part, T^* is polaroid, or equivalently T is polaroid. ∎

The following example provides an operator that is left polaroid but not a-polaroid.

Example 6.10. Let $R \in \ell^2(\mathbb{N})$ be the unilateral right shift defined as

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \cdots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}),$$

and

$$U(x_1, x_2, \dots) := (0, x_2, x_3, \cdots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

If $T := R \oplus U$ then $\sigma(T) = D(0,1)$, so iso $\sigma(T) = \emptyset$. Moreover, $\sigma_{\rm ap}(T) = \Gamma \cup \{0\}$, Γ the unit circle, so iso $\sigma_{\rm ap}(T) = \{0\}$. Since R is injective and $p(U) = 1$ it then follows that $p(T) = p(R) + p(U) = 1$. Furthermore, $T \in \Phi_+(X)$ and hence $T^2 \in \Phi_+(X)$, so that $T^2(X)$ is closed. Therefore 0 is a left pole and hence T is left polaroid. On the other hand $q(R) = \infty$, so that $q(T) = q(R) + q(U) = \infty$, so T is not a-polaroid. Note that T is also polaroid.

The following example provides an operator that is polaroid but not left polaroid.

Example 6.11. Let R denote the right shift on $\ell^2(\mathbb{N})$ defined by

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \quad (x_n) \in \ell^2(\mathbb{N}),$$

and let Q be the weighted left shift defined by

$$Q(x_1, x_2, \dots) := (x_2/2, x_3/3, \dots) \quad (x_n) \in \ell^2(\mathbb{N}).$$

Q is a quasi-nilpotent operator, $\sigma(R) = D(0,1)$, where $D(0,1)$ denotes the closed unit disc of $\mathbb{C}$, and $\sigma_{\rm ap}(R) = \Gamma$, where Γ is the unit circle of $\mathbb{C}$. Moreover, if $e_n := (0, ..., 0, 1, 0...)$, where 1 is the n-th term, then $e_{n+1} \in ker\, Q^{n+1}$ while $e_{n+1} \notin ker\, Q^n$ for every $n \in \mathbb{N}$, so $p(Q) = \infty$.

Define $T := R \oplus Q$ on $X := \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$. Clearly, $\sigma(T) = D(0,1)$, and $\sigma_{\rm ap}(T) = \Gamma \cup \{0\}$. We have $p(T) = p(R) + p(Q) = \infty$, so 0 is not a left pole. Therefore, T is polaroid, since iso $\sigma(T) = \emptyset$, but not left polaroid. Evidently, the dual T^* is polaroid but not right polaroid, since $q(T^*) = \infty$.

In the case of Hilbert space operators $T \in L(H)$ instead of the dual T^* it is more appropriate to consider the Hilbert adjoint T'. By means of the classical Fréchet- Riesz representation theorem we know that if U is the conjugate-linear isometry that associates to each $y \in H$ the linear form $x \to \langle x, y \rangle$ then

$$\overline{\lambda} I - T' = (\lambda I - T)' = U^{-1}(\lambda I - T)^* U. \tag{24}$$

This obviously implies that $\sigma_{\rm ap}(T') = \overline{\sigma_{\rm ap}(T^*)}$ and $\sigma_{\rm s}(T') = \overline{\sigma_{\rm s}(T^*)}$.

Theorem 6.12. *[4] If $T \in L(H)$, H a Hilbert space, then the following equivalences hold:*

(i) *T is left polaroid if and only if T' is right polaroid.*

(ii) *T is right polaroid if and only if T' is left polaroid.*

(iii) *T is polaroid if and only if T' is polaroid.*

In presence of SVEP the polaroid conditions coincide. Precisely, we have

Theorem 6.13. *[4] Let $T \in L(X)$. Then we have*

(i) *If T^* has SVEP at at every $\lambda \notin \sigma_{\rm uw}(T)$ then $\sigma_{\rm ap}(T) = \sigma(T)$. Furthermore, the properties of being polaroid, a-polaroid and left polaroid for T are all equivalent.*

(ii) *If T has SVEP at every $\lambda \notin \sigma_{\rm lw}(T)$ then $\sigma_{\rm s}(T) = \sigma(T)$. Furthermore,the properties of being polaroid, a-polaroid and left polaroid for T^* are all equivalent.*

Let $\mathcal{H}_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is nonconstant on each of the components of its domain. Define, by the classical functional calculus, $f(T)$ for every $f \in \mathcal{H}_{nc}(\sigma(T))$.

Theorem 6.14. *[3] For an operator $T \in L(X)$ the following statements are equivalent.*

(i) T is polaroid;

(ii) $f(T)$ is polaroid for every $f \in \mathcal{H}_{nc}(\sigma(T))$;

(iii) there exists a non-trivial polynomial p such that $p(T)$ is polaroid;

(iv) there exists $f \in \mathcal{H}_{nc}(\sigma(T))$ such that $f(T)$ is polaroid.

Theorem 6.15. *[18] If $T \in L(X)$ and $f \in \mathcal{H}_{nc}(\sigma(T))$, then λ is a left pole (respectively, right pole) of $f(T)$ if and only if there exists a left pole (respectively, a right pole) μ of T such that $f(\mu) = \lambda$.*

Proof. We show that $f(\Gamma_l(T)) = \Gamma_l(f(T))$. It is well-known that the spectral mapping theorem holds for $\sigma_{ap}(T)$. Moreover, $\Gamma_l(T)$ and $\sigma_{ld}(T)$ are disjoint. We have

$$f(\sigma_{ap}(T)) = f(\Gamma_l(T) \cup \sigma_{ld}(T)) = f(\Gamma_l(T)) \cup f(\sigma_{ld}(T)).$$

On the other hand,

$$\sigma_{ap}(f(T)) = \Gamma_l(f(T)) \cup \sigma_{ld}(f(T)) = \Gamma_l(f(T)) \cup f(\sigma_{ld}(T)),$$

and since $f(\sigma_{ap}(T)) = \sigma_{ap}(f(T))$ it then follows that $f(\Gamma_l(T)) = \Gamma_l(f(T))$.

The case of right poles may be proved in a similar way. To prove that $f(\Gamma_r(T)) = \Gamma_r(f(T))$, just replace $\Gamma_l(T)$ by $\Gamma_r(T)$ and $\sigma_{ap}(T)$ by $\sigma_s(T)$. ∎

A natural question is if the analogous of Theorem 6.14 holds for left polaroid operators. By using the same arguments of the proof of Theorem 6.14 (just use the spectral mapping theorem for $\sigma_{ap}(T)$ and $\sigma_s(T)$) we easily obtain that the implication

$$T \text{ left polaroid} \Rightarrow f(T) \text{ left polaroid},$$

holds for every $f \in \mathcal{H}_{nc}(\sigma(T))$, and a a similar implication holds also for right polaroid operators.

Denote by $\mathcal{H}^i_{nc}(\sigma(T))$ the set of all $f \in \mathcal{H}_{nc}(\sigma(T))$ such that f is injective.

Theorem 6.16. *[3] For an operator $T \in L(X)$ the following statements are equivalent.*

(i) T is left polaroid;

(ii) $f(T)$ is left polaroid for every $f \in \mathcal{H}^i_{nc}(\sigma(T))$;

(iii) there exists $f \in \mathcal{H}^i_{nc}(\sigma(T))$ such that $f(T)$ is left polaroid.

Proof. We have only to show that (iii) ⇒ (i). Let λ_0 be an isolated point of $\sigma_{ap}(T)$ and let $\mu_0 := f(\lambda_0)$ As in the proof of Theorem 6.14 it then follows that $\mu_0 \in \operatorname{iso}\sigma_{ap}(f(T))$, so μ_0 is a left pole of $f(T)$. Now, by Theorem 6.15 there exists a left pole η of T such that $f(\eta) = \mu_0$ and since f is injective then $\eta = \lambda_0$. Therefore, T is left polaroid. ∎

Definition 6.17. *An operator $T \in L(X)$ is said to be* hereditarily polaroid *if every part of T is polaroid.*

A simple example shows that polaroid operator need not be necessarily hereditarily polaroid. Let $T := R \oplus Q$ on $H \oplus H$, where $H := \ell^2(\mathbb{N})$, R is the right shift and Q is quasi-nilpotent. Then $\sigma(T)$ is the unit disc, so $\operatorname{iso}\sigma(T)$ is empty and hence T is polaroid. On the other hand, if $M := \{0\} \oplus \ell^2(\mathbb{N})$, then $T|M$ is not polaroid, since Q is not polaroid. It is easily seen that the property of being hereditarily polaroid is similarity invariant, but is not preserved by a quasi-affinity.

The class of hereditarily polaroid operators is substantial; it contains several important classes of operators. The first class that we consider is the following one introduced by Oudghiri [79].

Definition 6.18. *A bounded operator $T \in L(X)$ is said to* belong to the class $H(p)$ *if there exists a natural $p := p(\lambda)$ such that:*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathbb{C}. \tag{25}$$

The property $H(p)$ is inherited by the restrictions on closed invariant subspaces:

Theorem 6.19. *Let $T \in L(X)$ be a bounded operator on a Banach space X. If T has the property $H(p)$ and Y is a closed T-invariant subspace of X then $T|Y$ has the property $H(p)$.*

Proof. If $H_0(\lambda I - T) = \ker(\lambda I - T)^p$ then

$$H_0((\lambda I - T)|Y) \subseteq \ker(\lambda I - T)^p \cap Y = \ker((\lambda I - T)|Y)^p,$$

from which we obtain $H_0((\lambda I - T)|Y) = \ker((\lambda I - T)|Y)^p$. ∎

The following result is an easy consequence of Theorem 6.19.

Corollary 6.20. *Every $H(p)$-operator T is hereditarily polaroid.*

The next result shows that property $H(p)$ is preserved by quasi-affine transforms.

Theorem 6.21. *If $S \in L(Y)$ has property $H(p)$ and $T \prec S$, then T has property $H(p)$*

Proof. We consider the case that $p := (\lambda I - T) = 1$ for all $\lambda \in \mathbb{C}$. Suppose S has property $H(1)$, i. $SA = AT$, with A injective. If $\lambda \in \mathbb{C}$ and $x \in H_0(\lambda I - T)$ then

$$\|(\lambda I - S)^n Ax\|^{1/n} = \|A(\lambda I - T)^n x\|^{1/n} \leq \|A\|^{1/n} \|(\lambda I - T)^n x\|^{1/n},$$

from which it follows that $Ax \in H_0(\lambda I - S) = \ker(\lambda I - S)$. Hence $A(\lambda I - T)x = (\lambda I - S)Ax = 0$ and, since A is injective, this implies that $(\lambda I - T)x = 0$, i.e. $x \in \ker(\lambda I - T)$. Therefore $H_0(\lambda I - T) = \ker(\lambda I - T)$ for all $\lambda \in \mathbb{C}$. The more general case of $H(p)$-operators is proved by a similar argument. ∎

The class of $H(p)$-operators is very large. To see this, we first introduce a special class of operators which has an important role in local spectral theory. Let $C^\infty(\mathbb{C})$ denote the Fréchet algebra of all infinitely differentiable complex-valued functions on $\mathbb{C}$

Definition 6.22. *An operator $T \in L(X)$, X a Banach space, is said to be* generalized scalar *if there exists a continuous algebra homomorphism $\Psi : C^\infty(\mathbb{C}) \to L(X)$ such that $\Psi(1) = I$ and $\Psi(Z) = T$, where Z denotes the identity function on $\mathbb{C}$.*

The interested reader can be find a a well organized treatment of generalized scalar operators in [68, Section 1.5]). It should be noted that every quasi-nilpotent generalized scalar operator is nilpotent, [68, Proposition 1.5.10]. Moreover, if T is generalized scalar then T has the Dunford property (C), i.e. $\mathcal{X}_T(\Omega)$ is closed for all closed subset $\Omega \subseteq \mathbb{C}$, see [68, Theorem 1.5.4 and Proposition 1.4.3]. In particular, $H_0(\lambda I - T) = \mathcal{X}_T(\{\lambda\}$ is closed for each $\lambda \in \mathbb{C}$, so every generalized scalar operator has SVEP, by Theorem 3.8.

An operator similar to a restriction of a generalized scalar operator to one of its closed invariant subspaces is called *subscalar*.

Theorem 6.23. *Every subscalar operator $T \in L(X)$ is $H(p)$.*

Definition 6.24. *An operator $T \in L(X)$ is said* paranormal *if*

$$\|Tx\| \leq \|T^2 x\| \|x\| \quad \textit{for all } x \in X,$$

The restriction $T|M$ of a paranormal operator $T \in L(X)$ to a closed subspace M is evidently paranormal . The property of being paranormal is not translation-invariant. $T \in L(X)$ is called *totally paranormal* if $\lambda I - T$ is paranormal for all $\lambda \in \mathbb{C}$.

Theorem 6.25. *Every totally paranormal operator has property $H(1)$.*

Theorem 6.23 implies that some important classes of operators are $H(p)$. In the sequel we list some of these classes:

(a) *Hyponormal operators.* A bounded operator $T \in L(H)$ on a Hilbert space is said to be hyponormal

$$\|T'x\| \leq \|Tx\| \quad \text{for all } x \in H,$$

or equivalently $T'T \geq TT'$. By an important result due to Putinar [80], every hyponormal operator is similar to a subscalar operator, see also [68, section 2.4], so hyponormal operators are $H(p)$. Actually, we have more: every hyponormal operator is $H(1)$ since, as it is easy to verify, is also totally paranormal. An example of hyponormal operator on $\ell^2(\mathbb{N})$ is given by a weighted right shift where the corresponding weight sequence is increasing. Subnormal operators and quasi-normal operators are hyponormal, see [39] or [50], so these operators are $H(1)$.

For $T \in L(H)$ let $T = W|T|$ be the polar decomposition of T. Then $R := |T|^{1/2}W|T|^{1/2}$ is said the *Aluthge transform* of T. If $R = V|R|$ is the polar decomposition of R, define $\widetilde{T} := |R|^{1/2}V|R|^{1/2}$.

(b) *Log-hyponormal operators.* An operator $T \in L(H)$ is said to be log-hyponormal if T is invertible and satisfies

$$\log(T^*T) \geq \log(TT^*).$$

If T is log-hyponormal then $\widetilde{T}$ is hyponormal and $T = K\widetilde{T}K^{-1}$, where $K := |R|^{1/2}|T|^{1/2}$, see ([95], [37]). Hence T is similar to a hyponormal operator and therefore, by Theorem 6.21, has property $H(1)$.

(c) *p-hyponormal operators.* An operator $T \in L(H)$ is said to be be p-hyponormal, with $0 < p \leq 1$, if

$$(T'T)^p \geq (T')^p.$$

Every invertible p-hyponormal T is quasi-similar to a log-hyponormal operator and consequently, by Theorem 6.21, it has property $H(1)$(([20], [40]). This is also true for p-hyponormal operators which are not invertible. Every p-hyponormal operator is paranormal, see [21] or [36].

(d) *M-hyponormal operators.* Recall that $T \in L(H)$ is said to be *M-hyponormal* if there exists $M > 0$ such that

$$TT^* \leq MT^*T.$$

Every M-hyponormal operator is subscalar ([68, Proposition 2.4.9]) and hence $H(p)$.

(e) *w-hyponormal operators.* If $T \in L(H)$ and $T = U|T|$ is the polar decomposition, define $\hat{T} := |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. $T \in L(H)$ is said to be w-hyponormal if

$$|\hat{T}| \geq |T| \geq |\hat{T}^*|.$$

Examples of w-hyponormal operators are p-hyponormal operators and log-hyponormal operators. All w-hyponormal operators are subscalar (together with its Aluthge transformation, see [71]), and hence $H(p)$.

(f) Another important class of polaroid operators is given by the class of all multipliers on a semi-simple commutative Banach algebra. Given a Banach algebra A, a map $T : A \to A$ is said to be a *multiplier* if $(Tx)y = x(Ty)$ holds for all $x, y \in \mathcal{A}$. For every multiplier of a semi-simple Banach algebra A we have $H_0(\lambda I - T) = \ker(\lambda I - T)$ for all $\lambda \in \mathbb{C}$, see [7, Theorem 1.8], so every multiplier is $H(1)$. A very important example of multiplier is given in the case where A is the semi-simple Banach algebra $L^1(G)$, the group algebra of a locally compact abelian group G with convolution as multiplication. Indeed, in this case to any complex Borel measure μ on G there corresponds a multiplier T_μ defined by

$$T_\mu(f) := \mu \star f \quad \text{for all } f \in L^1(G),$$

where

$$(\mu \star f)(t) := \int_G f(t-s)\,d\mu(s).$$

The classical Helson-Wendel Theorem shows that each multiplier is a convolution operator and the multiplier algebra of $A := L^1(G)$ may be identified with the measure algebra $M(G)$, see [67, Chapter 0].

Theorem 6.26. *([11, p. 2445] and [3]) Every paranormal operator on a separable Banach space has SVEP. Paranormal operators on Hilbert spaces have SVEP. Moreover, every algebraic paranormal operator $T \in L(X)$ is hereditarily polaroid*

The class of paranormal operators includes some other classes of operators defined on Hilbert spaces:

(g) *p-quasihyponormal operators.* In Example (c) it has been observed that every p-hyponormal operator is paranormal. A Hilbert space operator $T \in L(H)$ is said to be p-quasihyponormal for some $0 < p \leq 1$ if

$$T'|T'|^{2p}T \leq T'|T|^{2p}T.$$

Every p-quasi-hyponormal is paranormal [69].

(h) *Class A operators* An operator $T \in L(H)$ is said to be a class A operator if $|T^2| \geq |T|^2$. Every log-hyponormal operator is a class A operator [49] but the converse is no true, see [50, p. 176]. Every class A operator is paranormal (an example of a paranormal operator which is not a class A operator can be found in [50, p. 177]).

References

[1] P. Aiena *Fredholm and local spectral theory, with application to multipliers.* (2004), Kluwer Acad. Publishers.
[2] P. Aiena: *Quasi Fredholm operators and localized SVEP*. Acta Sci. Math. (Szeged), **73** (2007), 251–63.
[3] P. Aiena: *Algebraically paranormal operators on Banach spaces*. Banach J. Math. Anal. 7 (2013), no. 2, 136145.
[4] P. Aiena, E. Aponte: *Polaroid type operators under perturbations.* Studia Math. **214** (2013), (2), 121-136.
[5] P. Aiena, M. T. Biondi, C. Carpintero *On Drazin invertibility*. Proc. Amer. Math. Soc. **136**, (2008), 2839—48.
[6] P. Aiena, C. Carpintero, E. Rosas *Some characterization of operators satisfying a-Browder theorem*. J. Math. Anal. Appl. **311**, (2005), 530–44.
[7] P. Aiena, M.L. Colasante, M. González *Operators which have a closed quasi-nilpotent part.* Proc. Amer. Math. Soc. **130**, (9), 2701–10.
[8] P. Aiena, T. L. Miller, M. M. Neumann *On a localized single-valued extension property*. Math. Proc. Royal Irish Ac. **104A** (1), (2004), 17-34.
[9] P. Aiena, M. Chō, M. González: *Polaroid type operator under quasi-affinities.* J. Math. Anal. Appl. 371, (2010), no. 2, 485–495.
[10] P. Aiena, M. Gonzlez *On the Dunford property (C) for bounded linear operators RS and SR*. Integr. Equa. Oper. Theory 70 (2011), no. 4, 56158.
[11] P. Aiena, J. R. Guillen *Weyl's theorem for perturbations of paranormal operators*. Proc. Amer. Math. Soc.**35**, (2007), 2433–42.
[12] P. Aiena, J. Guillen, P. Peña *Property (gb) through local spectral theory*. Mathematical Proceedings of the Royal Irish Academy **114A** (2014), 1-15.
[13] P. Aiena, O. Monsalve *Operators which do not have the single valued extension property.* J. Math. Anal. Appl. **250** n. 2, (2003), 435–48.
[14] P. Aiena, O. Monsalve *The single valued extension property and the generalized Kato decomposition property.* Acta Sci. Math.(Szeged) **67** (2001), 461-477.
[15] P. Aiena, V. Muller *The localized single-valued extension property and Riesz operators.* (2014), to appear in Proc. Amer. Math. Soc.
[16] P. Aiena, M. M. Neumann: *On the stability of the localized single-valued extension property under commuting perturbations*. Proc. Amer. Math. Soc. **141** (2013), no. 6, 20392050.
[17] P. Aiena, E. Rosas *The single valued extension property at the points of the approximate point spectrum*, J. Math. Anal. Appl. **279** (1), (2003), 180–88.
[18] P. Aiena, J. E. Sanabria: *On left and right poles of the resolvent*. Acta Sci. Math. (Szeged), **74**, (2008), 669-87.
[19] P. Aiena, S. Triolo: *Some perturbation results through localized SVEP*. (2014), to appear in Acta Sci. Math. (Szeged).
[20] A. Aluthge, (1990). On p-hyponormal operators for $1 < p < 1$., *Integr. Equa. Oper Theory* **13**, 307–15.
[21] T. Ando. *Operators with a norm condition*, Acta Sci. Math. (Szeged) **33**, (1972), 169-178.
[22] B.A. Barnes, (1998). *Common operator properties of the linear operators RS and SR.* Proc. Amer. Math. Soc., vol. **126**, no. 4, 1055–61.
[23] C, Benhida, E.H. Zerouali *Local spectral theory of linear operators RS and SR.* Integr. Equa. Oper. Theory **54** (2006), 1-8.
[24] M. Berkani *On a class of quasi-Fredholm operators*. Int. Equa. Oper. Theory **34**, (1), (1999), 244–49.
[25] M. Berkani *Restriction of an operator to the range of its powers*. Studia Math. **140** (2), (2000), 163–75.
[26] M. Berkani, M. Sarih *On semi B-Fredholm operators*. Glasgow Math. J. **43**, (2001), 457–465.
[27] M. Berkani *Index of B-Fredholm operators and generalization of a Weyl's theorem*. Proc. Amer. Math. Soc., vol. 130, **6**, (2001), 1717–23.
[28] M. Berkani *B-Weyl spectrum and poles of the resolvent*. J. Math. Anal. Appl. **272**, (2002), 596–603.
[29] M. Berkani, J. J. Koliha *Weyl type theorems for bounded linear operators*. Acta Sci. Math. (Szeged) **69**, (2003), no. 1-2, 359–76.
[30] M. Berkani, M. Sarih *On semi B-Fredholm operators*. Glasgow Math. J. **43**, (2001), 457–465.
[31] M. Berkani, M. Sarih *An Atkinson-type theorem for B-Fredholm operators*. Studia Math. **148** (2001), no. 3, 251–57.
[32] E. Bishop, (1959). A duality theorem for an arbitrary operator., *Pacific J. Math.* **9**, 379–97.
[33] A. Bourhim, V: G. Miller *The single valued extension property is not preserved under sums and products of commuting operators* Glasgow Math. J. **49** (2007), 99-104.
[34] S. R. Caradus, W.E. Pfaffenberger, B. Yood *Calkin algebras and algebras of operators in Banach spaces.* (1974), *Dekker, New York*.
[35] S. R. Caradus *Operator theory of the pseudo-inverse.* Queen's Papers in Pure and Appl. Math **38**,(1974) Queen's Univ. Kingston.

[36] M. Chō. *Spectral properties of p-hyponormal operators* Glasgow Math. J. **436** (1994), 117-122.
[37] M. Chō, I.H. Jeon, J.I. Lee, *Spectral and structural properties of log-hyponormal operators.* Glasgow Math. J. **42**, (2000), 345–50.
[38] I. Colojoară, C. Foiaş *Theory of generalized spectral operators.*, (1968), *Gordon and Breach, New York.*
[39] J. B. Conway *Subnormal operators* Michigan Math. J. **20**, (1970), 529-544.
[40] B.P. Duggal, S V. Djordjević *Weyl's theorems and continuity of the spectra in the class of p-hyponormal operators.* Studia Math. **143**, no. 1, (2000), 23–32.
[41] B. P. Duggal, H. Jeon. *Remarks on spectral properties of p-hyponormal and log-hyponormal operators.* Bull. Kor. Math. Soc. **42**, (2005), 541–52.
[42] M. P. Drazin: *Pseudoinverse in associative rings and semigroups.* Amer. Math. Monthly **65**, (1958), 506-514.
[43] N. Dunford *Spectral theory II. Resolution of the identity.* Pacific J. Math. **2**, (1952), 559–614.
[44] N. Dunford *Spectral operators.* Pacific J. Math. **4**, (1954), 321–54.
[45] N. Dunford *A survey of the theory of spectral operators.*, Bull. Amer. Math. Soc. **64**, , (1958), 217–74.
[46] N. Dunford, J.T. Schwartz *Linear operators.* Part I (1967), Part II (1967), Part III, Wiley, New York.
[47] P. A. Filmore, J. P. Williams *On operator ranges.* Advances in Math. **7**, , (1971), 254–81.
[48] J. K. Finch *The single valued extension property on a Banach space* Pacific J. Math. **58** (1975), 61-69.
[49] T. Furuta, M. Ito and T. Yamazaki *A subsclass of paranormal operators including class of log-hyponormal and several related classes.*, Scientiae Mathematicae **1** (1998), 389-403.
[50] T. Furuta. *Invitation to linear operators.*, Taylor and Francis, London-N.York 2001.
[51] M. González *The fine spectrum of the Cesàro operator in* ℓ_p, $(1 < p < \infty)$. Arch. Math. **44** (1985), 355–358.
[52] M. González,M. Mbekhta and M. Oudghiri *On the isolated points of the surjective spectrum of a bounded operator.* Proc. Amer. Math. Soc. **136** (2008), no. 10, 3521-28.
[53] S. Grabiner *Ranges of products of operators.* Canad. J. Math. **26**, (1974), 1430–41.
[54] S. Grabiner *Uniform ascent and descent of bounded operators.* J. Math. Soc. Japan **34**, 2, (1982), 317–37.
[55] R.E. Harte *Fredholm, Weyl and Browder theory.* Proc. Royal Ir. Acad. **85A**, No. 2, (1985), 151–76.
[56] R.E. Harte *Fredholm, Weyl and Browder theory II.* Proc. Royal Ir. Acad. **91A**, No. 1, , (1991), 79–88.
[57] R.E. Harte *Invertibility and singularity for bounded linear operators.* (1988), *Marcel Dekker*, Vol. 109.
[58] H. Heuser *Functional Analysis.*, (1982), Wiley Interscience, Chichester.
[59] J.R. Holub *On shifts operators,* Canad. Math. Bull. **31**, (1988), 85–94.
[60] Q. Jiang, H. Zhong *Generalized decomposition, single-valued extension property and approximate point spectrum* J. Math. Anal. Appl. **356**, (2009), 322–7.
[61] Q. Jiang, H. Zhong *Topological uniform descent and localized SVEP* J. Math. Anal. Appl. **390**, (2012), 355–61.
[62] R. V. Kadison, J. R. Ringrose *Fundamentals of the theory of operator algebras* Vol. I , Academic Press, New York, 1983.
[63] T. Kato *Perturbation theory for nullity, deficiency and other quantities of linear operators.* J. Anal. Math. **6**, (1958), 261–322.
[64] T. Kato *Perturbation theory for linear operators.* (1966), Springer-Verlag, New York.
[65] V. Kordula, V. Müller, V. Rakočević *On the semi-Browder spectrum.* Studia Math. **123**, (1997), 1–13.
[66] J.P. Labrousse *Les opérateurs quasi-Fredholm.* Rend. Circ. Mat. Palermo, XXIX **2**, (1980), 161–258.
[67] R. Larsen *An introduction to the theory of multipliers.* Springer-Verlag, New York, 1979.
[68] K.B. Laursen, M.M. Neumann *An introduction to local spectral theory.* (2000), London Math. Soc. Monographs 20, Clarendon Press, Oxford.
[69] M. Y. Lee, S. H. Lee. Some generalized theorems on p-quasihyponormal operators for $0 < p < 1$. Nihonkai Math. J. **8**, (1997), 109-115.
[70] J. Leiterer *Banach coherent analytic Fréchet sheaves.* Math. Nachr. **85**, (1978), 91-109.
[71] C. Lin, Y. Ruan, Z. Yan. w-hyponormal operators are subscalar. Integr. Equ. oper. theory. **50**, (2004), 165-168.
[72] M. Mbekhta *Sur la théorie spectrale locale et limite des nilpotents.* Proc. Amer. Math. Soc. **110**, (1990), 621–31.
[73] M. Mbekhta *Local spectrum and generalized spectrum.*, Proc. Amer. Math. Soc. **112**, (1991), 457–63.
[74] M. Mbekhta, V. Müller *On the axiomatic theory of spectrum II.* Studia Math. **119**, (1996), 129–47.
[75] M. Mbekhta, A. Ouahab *Opérateur s-régulier dans un espace de Banach et théorie spectrale.* Acta Sci. Math. (Szeged) **59**, (1994), 525–43.
[76] V. Müller *On the regular spectrum.*, J. Operator Theory **31**, (1994), 363–80.
[77] V. Müller *On the Kato decomposition of quasi-Fredholm and B-Fredholm operators.*, Preprint ESI 1013, Vienna 2001.
[78] V. Müller *Spectral theory of linear operators and spectral systems on Banach algebras.* (2003), Birkh*ä*user Verlag, Berlin.
[79] M. Oudghiri *Weyl's and Browder's theorem for operators satisfying the SVEP.* Studia Math. **163**, 1, (2004), 85–101.
[80] M. Putinar *Hyponormal operators are subscalar* J. Operator Theory. **12**, (1984), 385–95.
[81] V. Rakočević Generalized spectrum and commuting compact perturbation., *Proc. Edinburgh Math. Soc.* (1993), **36** (2), 197–209.
[82] V. Rakočević*Semi-Browder operators and perturbations.* Studia Math. **122**, (1997), 131–7.
[83] B.E. Rhoades *Spectra of some Hausdorff operators.* Acta Sci Math. (Szeged) **32** (1971), 91–100.
[84] S. Roch, B. Silbermann *Continuity of generalized inverses in Banach algebras.* Studia Math. **136** (1999), 197–227.
[85] W. Rudin*Functional analysis.*,(1991), 2nd edn., McGraw-Hill, New York.
[86] P. Saphar *Contribution á l'étude des applications linéaires dans un espace de Banach.* Bull. Soc. Math. France **92**, (1964), 363–84.
[87] M. Schechter, R. Withley *Best Fredholm perturbation theorems.* Studia Math. **90**, 1, (1988), 175-190.
[88] C. Schmoeger *On a class of generalized Fredholm operators I.* Demonstratio Math. **30** (1997), 829-42.
[89] C. Schmoeger *On a class of generalized Fredholm operators II.* Demonstratio Math. **31** (1998), 70522.
[90] C. Schmoeger *On a class of generalized Fredholm operators III.* Demonstratio Math. **31** (1998), 723-33.
[91] C. Schmoeger *On a class of generalized Fredholm operators IV.* Demonstratio Math. 32 (1999), 595604.
[92] C. Schmoeger *On a class of generalized Fredholm operators V.* Demonstratio Math. **32** (1999),n 595-604

[93] C. Schmoeger, Christoph *On a class of generalized Fredholm operators VI*. Demonstratio Math. 32 (1999), no. 4, 811822.

[94] A. L. Shields *Weighted shift operators and analytic function theory.* (1974), Topics in Operator theory, Mathematical Survey N.**105** (ed. C. Pearcy), 49–128. Am. Math. Soc., Providence, RI.

[95] K. Tanahashi *On log-hyponormal operators.* Inte. Equa. Oper. Theory **34**, (1999), 364-72.

[96] F.-H. Vasilescu, *Analytic functional calculus and spectral decompositions,* Editura Academiei and D. Reidel Publishing Company, Bucharest and Dordrecht, 1982.

[97] P. Vrbová*On local spectral properties of operators in Banach spaces.* Czechoslovak Math. J. **23**(98), (1973a), 483–92.

[98] Q. Zeng, H. Zhong *Common properties of bounded linear operators AC and BA: Local spectral theory* J. Math. Anal. Appl. **414** (2), (2014), 553-60.